Graphs and cycles in curve-based collision walks

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Mathematics of Post-Quantum Cryptography

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TU/e

Three problems

The ECDLP problem

Input: Two points $P, Q \in E(\mathbb{F}_q)$ with $Q \in \langle P \rangle$.
Question: Find an integer x such that $xP = Q$.

The Isogeny Pathfinding Problem over $\mathbb{F}_{p^2}$

Input: Two supersingular elliptic curves E_0 and E_1 defined over $\mathbb{F}_{p^2}$ on an ℓ-isogeny graph.
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Elliptic curves

What is an elliptic curve?

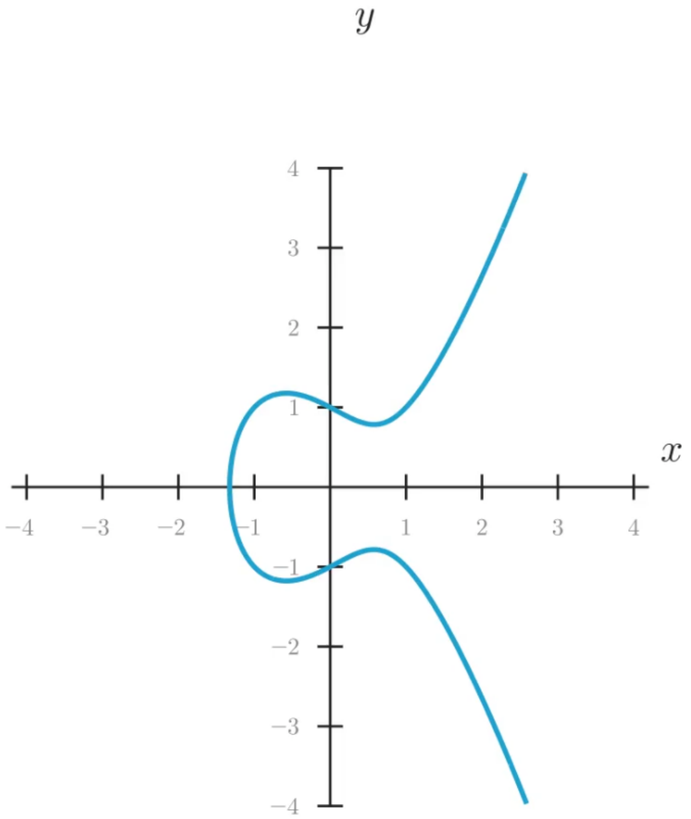
An elliptic curve is an algebraic curve that admits an affine equation of the form

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

(general form of a Weierstrass curve)

with $a_i \in k$, where k is the field where the point is defined.

Example. $y^2 = x^3 - x + 1$



→ A point on E means that the point (x, y) satisfies the curve equation.

Elliptic curves over $\mathbb{F}_q$

In cryptography, we use elliptic curves over finite fields $\mathbb{F}_q$, $q = p^k$ (but we draw the figures over $\mathbb{R}$ because it's nicer).

→ We denote by $E(k)$ the set of k-rational points on E.

↳ defined over k

Hasse bound

$$\#E(\mathbb{F}_q) \in [q + 1 - 2\sqrt{(q)}, q + 1 + 2\sqrt{(q)}]$$

Elliptic curves in cryptography

- Short Weierstrass form

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- The set of points on E with the addition law form a group.

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Jacobi criterion: a point on E is singular if (x, y) also satisfies the two partial derivatives $2y + a_1x + a_3 = 0$ and $a_1y = 3x^2 + 2a_2x + a_4$.

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- supersingular (also non-singular) if and only if $\#E(\mathbb{F}_p) = p + 1$ (for $p > 3$).

  equivalently: iff $E[p] = \{\infty\}$

  $E[n] = \{P \in E(\overline{\mathbb{F}_p}) \mid nP = \infty\}$ (the n-torsion group)

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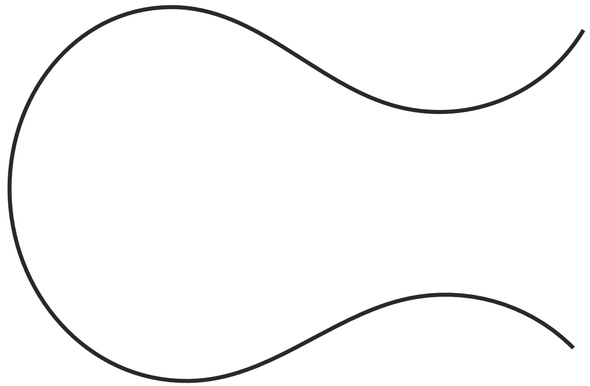
Supersingular curves and cyclic groups

- $E(\mathbb{F}_p) \cong \mathbb{Z}/(p+1)$
- $E(\mathbb{F}_{p^2}) \cong \mathbb{Z}/(p+1) \times \mathbb{Z}/(p+1)$

Isogenies

A familiar isogeny

The multiplication-by-d map



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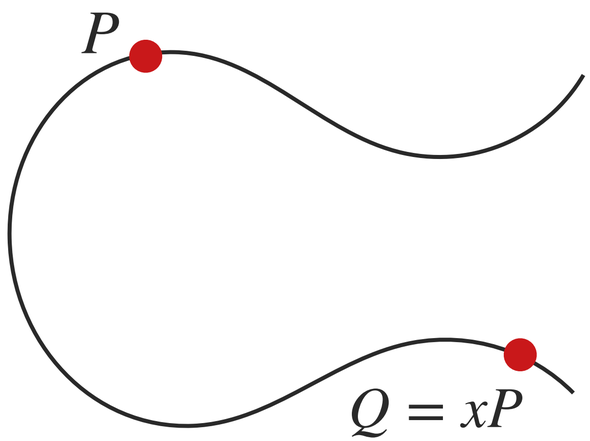

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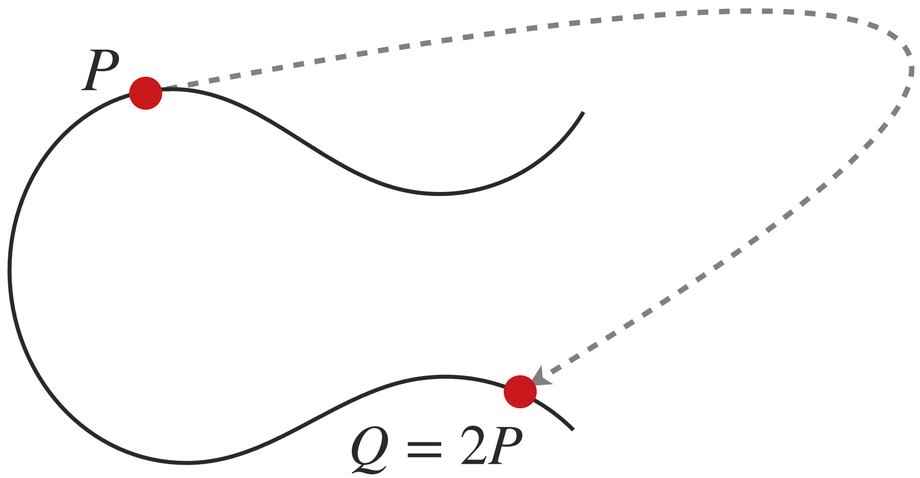
$$(x, y) \mapsto (\lambda^2 - 2x, \lambda x + y),$$

$$\lambda = \frac{3x^2 + a}{2y}$$



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j-invariant

- An invariant under isomorphisms.
- For a curve in short Weierstrass form $y^2 = x^3 + c_4 x + c_6$, we have

$$j = 1728 \cdot 4c_4^3/(4c_4^3 + 27c_6^2)$$

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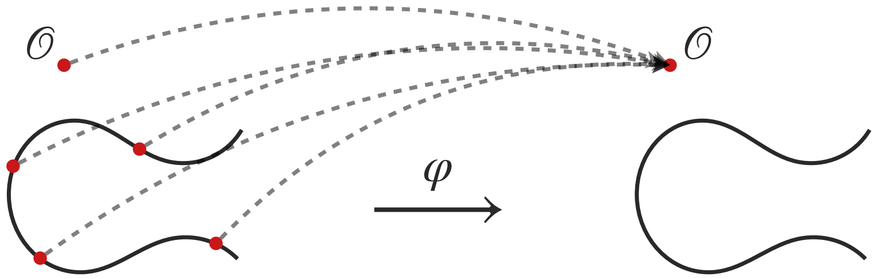

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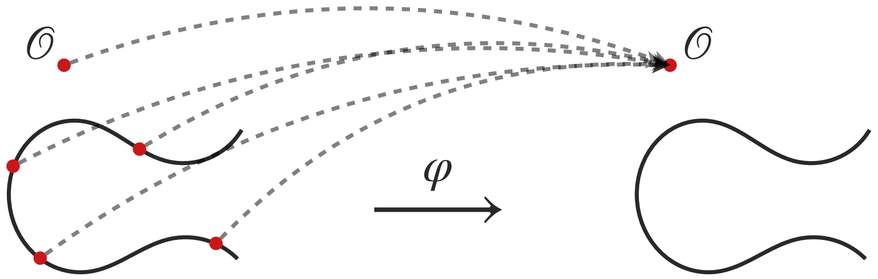
→ The degree of a (separable) isogeny is the size of its kernel.



Isogenies

Example.

$$(x, y) \mapsto \left(\frac{x^3 - 4x^2 + 30x - 12}{(x-2)^2}, \frac{x^3 - 6x^2 - 14x + 35}{(x-2)^3} \cdot y\right)$$

defines a degree-3 isogeny of the elliptic curves

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over $\mathbb{F}_{71}$. Its kernel is $\{(2,9), (2, -9), \mathcal{O}\}$.

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ℓ-isogeny:

- $x \to \frac{f(x)}{g(x)}$, with $\deg(f) = \ell, \deg(g) = \ell - 1$
- $y \to \ldots$

Computing isogenies

*We consider only supersingular curves from now on.

→ **Goal**: Compute an ℓ-isogeny from E.

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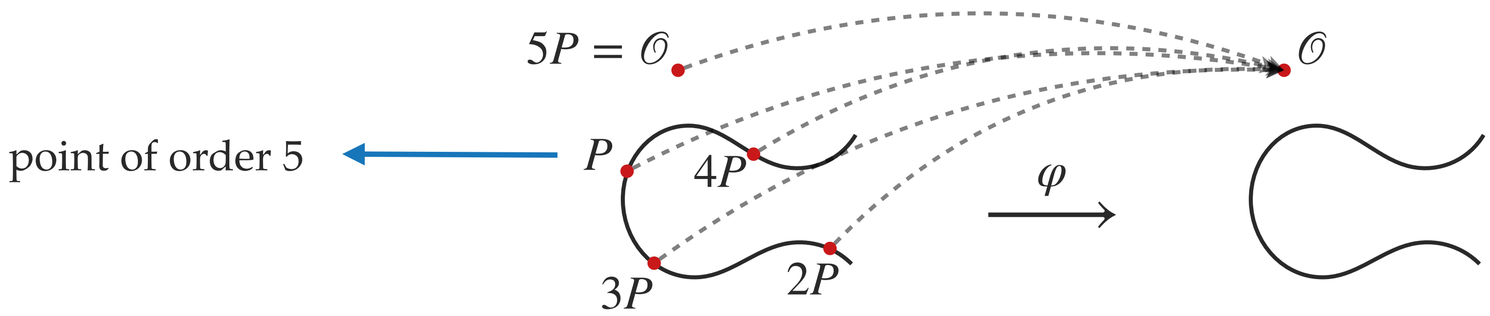

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Let P have prime order ℓ on E_A.
For $1 \leq i < \ell$ let x_i be the x-coordinate of iP.
Let

$$\tau = \prod_{i=1}^{\ell-1} x_i, \quad \sigma = \sum_{i=1}^{\ell-1} \left(x_i - \frac{1}{x_i} \right), \quad f(x) = x \prod_{i=1}^{\ell-1} \frac{x x_i - 1}{x - x_i}.$$

Then the ℓ-isogeny with kernel $\langle P \rangle$ is given by

$$\varphi_\ell : E_A \to E_B, (x, y) \mapsto (f(x), c_0 y f'(x))$$

where $B = \tau(A - 3\sigma)$, and $c_0^2 = \tau$.

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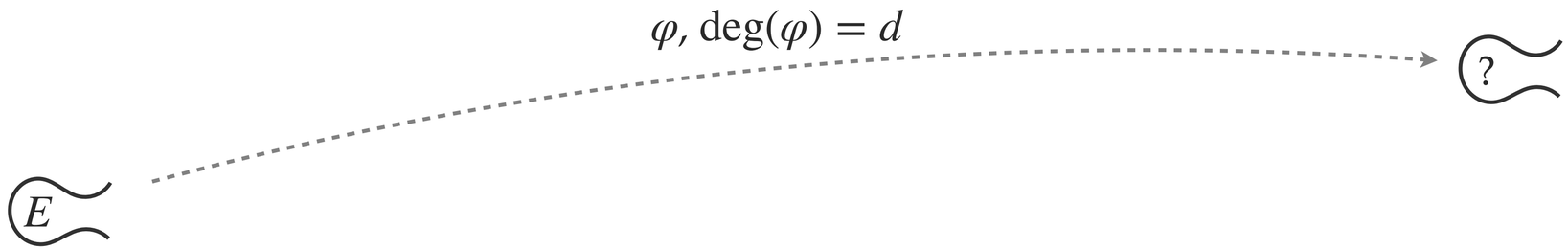

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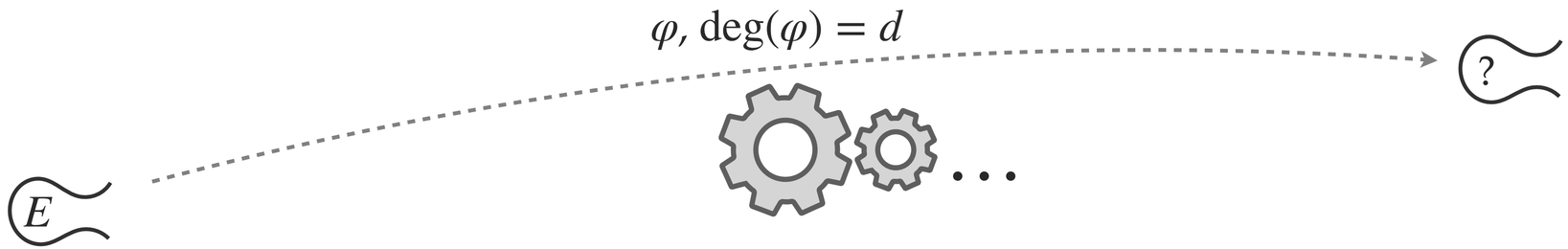
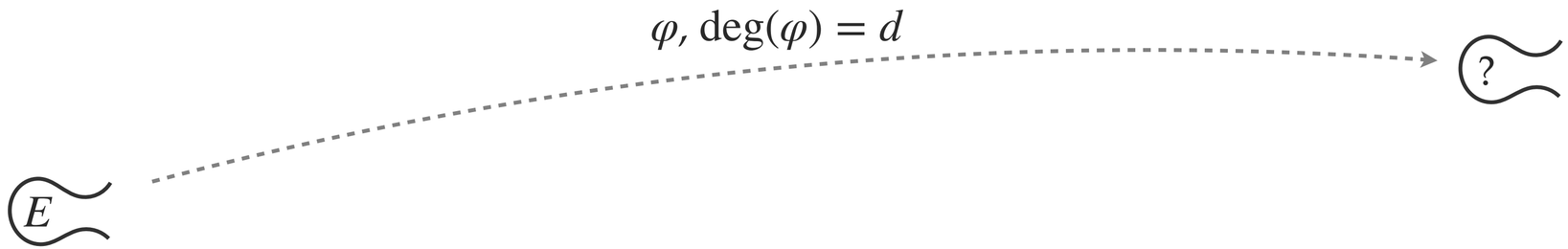


Composing isogenies

Goal: Compute a d-isogeny from E, with d a smooth integer ($d = \ell_1^{e_1} \cdot \ell_2^{e_2} \cdots \ell_n^{e_n}$).



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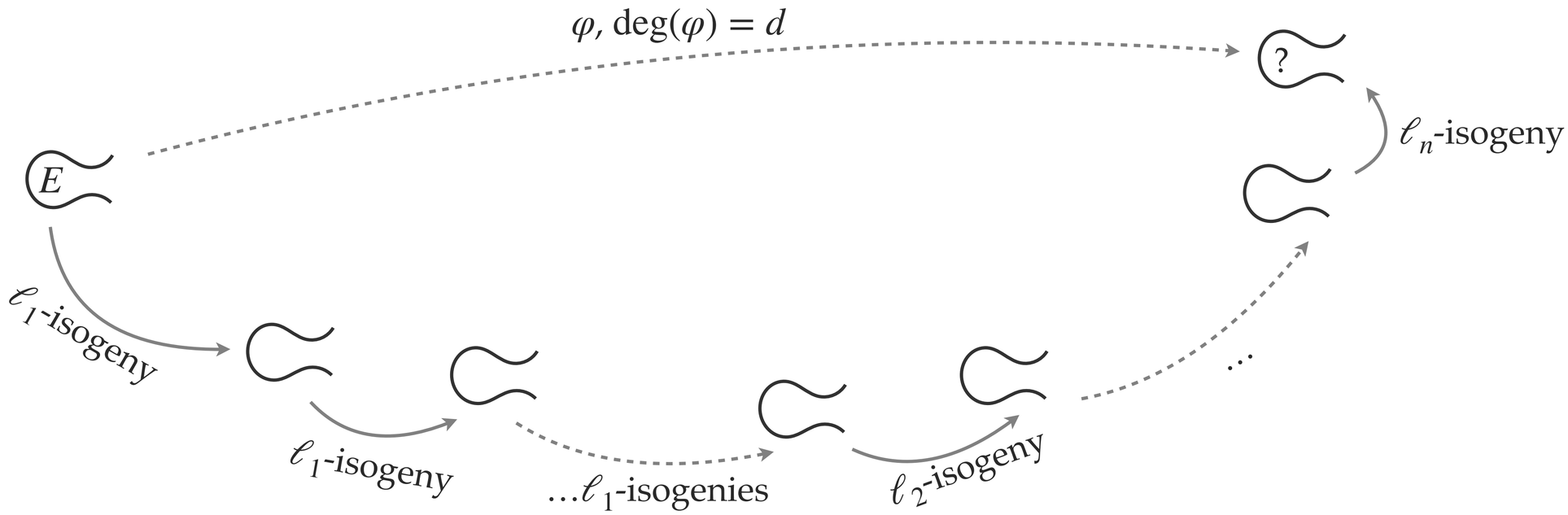

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$\mathrm{End}(E)$

- An endomorphism is an isogeny from a curve E to itself.
- The set of endomorphisms forms a ring $\mathrm{End}(E)$ under $+$ and $\circ$.

Hard problems and reductions

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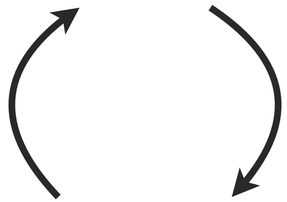

The EndRing problem

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Isogeny graphs

Supersingular curves and cyclic groups (recall)

- $E(\mathbb{F}_p) \cong \mathbb{Z}/(p+1)$
- $E(\mathbb{F}_{p^2}) \cong \mathbb{Z}/(p+1) \times \mathbb{Z}/(p+1)$

Let ℓ be a prime factor of p.

There are exactly $(\ell+1)$ $\mathbb{F}_{p^2}$-rational ℓ-isogenies from each E.

There is exactly one $\mathbb{F}_p$-rational ℓ-isogeny from each E.

Isogeny graphs

- Vertices are (isomorphism classes) of supersingular elliptic curves.
- Edges are prime-degree isogenies between them.

The vOW framework

Three problems for van Oorschot and Wiener

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The Isogeny Pathfinding Problem over $\mathbb{F}_p$

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The multi-user setting

→ We have many users, say L, with the same system parameters.

Example. (ECDLP)

User L_j has public key Q_j, secret key x_j such that $Q_j = x_j P$.

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User L_j has $\begin{matrix}\text{public key } Q_j,\\ \text{secret key } x_j\end{matrix}$ such that $Q_j = x_j P$.

Goal. If we can recover the key of one user in $\mathcal{O}($😺$)$,

we recover the key of all users in $< \mathcal{O}(L$ 😺$)$.

Another three problems for van Oorschot and Wiener

The MU ECDLP problem

Input: $L+1$ points $P, Q_1, \ldots, Q_L \in E(\mathbb{F}_q)$ with $Q_j \in \langle P \rangle$.
Question: Find integers $a_1, \ldots, a_L$ such that $Q_j = a_j P$ for $j \in \{1,\ldots,L\}$.

The MU Isogeny Pathfinding Problem over $\mathbb{F}_{p^2}$

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The MU Isogeny Pathfinding Problem over $\mathbb{F}_p$

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vOW for ECDLP

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and we compute

$$x = \frac{a - a'}{b' - b} \pmod{N}.$$

Collision search

Collision

Given a random map $f : S \to S$ on a finite set S of cardinality N, we call collision any pair R, R' of elements in S such that $f(R) = f(R')$.

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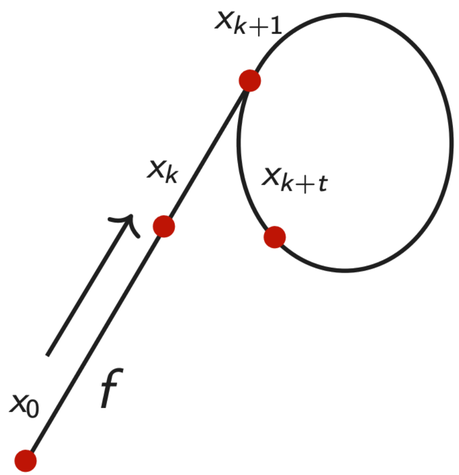
Pollard's Rho method



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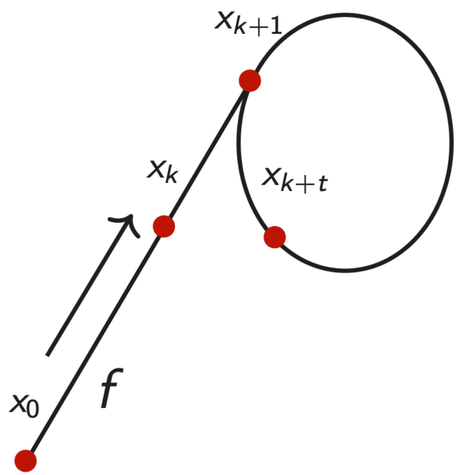

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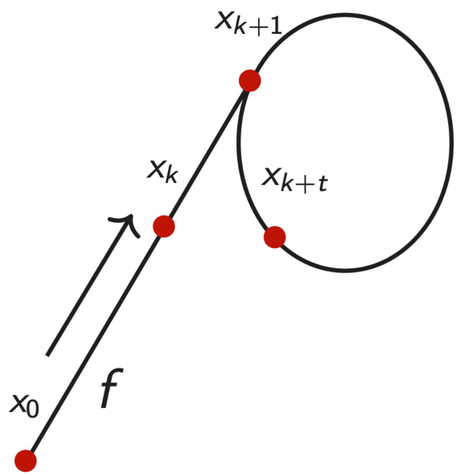

- Ideally, f is a random mapping.
- Expected number of steps until the collision is found

$$\sqrt{\frac{\pi N}{2}}$$

Collision search

$$f(R) = \begin{cases} R + c_1 P + d_1 Q & \text{if } R \in S_1 \\ R + c_2 P + d_2 Q & \text{if } R \in S_2 \\ \cdots & \\ R + c_k P + d_k Q & \text{if } R \in S_k, \end{cases}$$

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Property of f

Input $(aP + bQ) \rightarrow$ Output $(a'P + b'Q)$.
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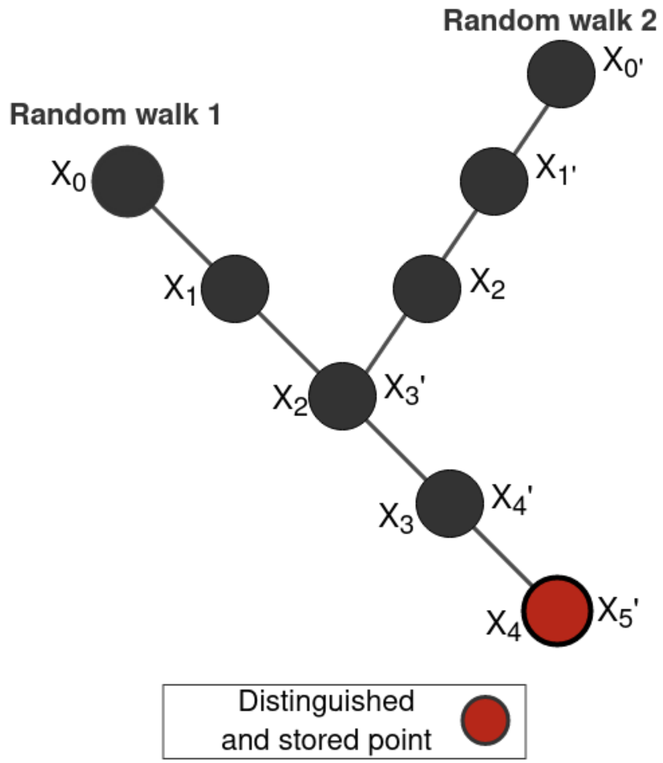
Intuitively:

- Start from $R = aP + bQ$ for some random a and b.
- *Walk* the random walk until we find the same point twice.
  ↪ To discover the collision, we need to store *all** the points that we compute.

Parallel Collision Search



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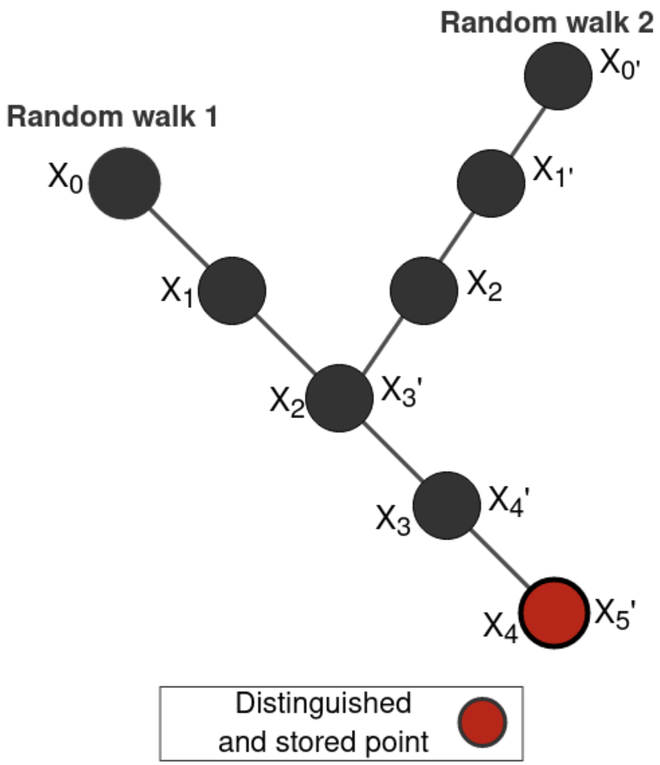

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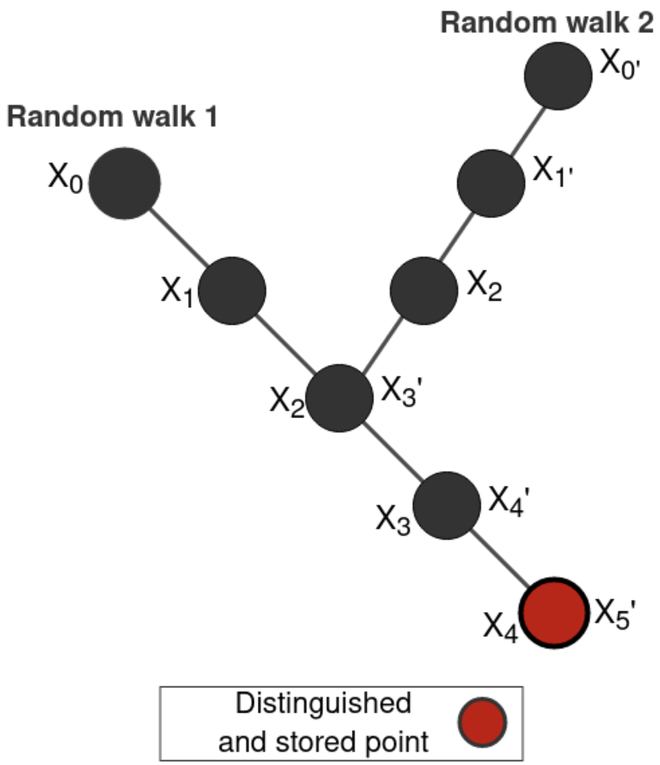

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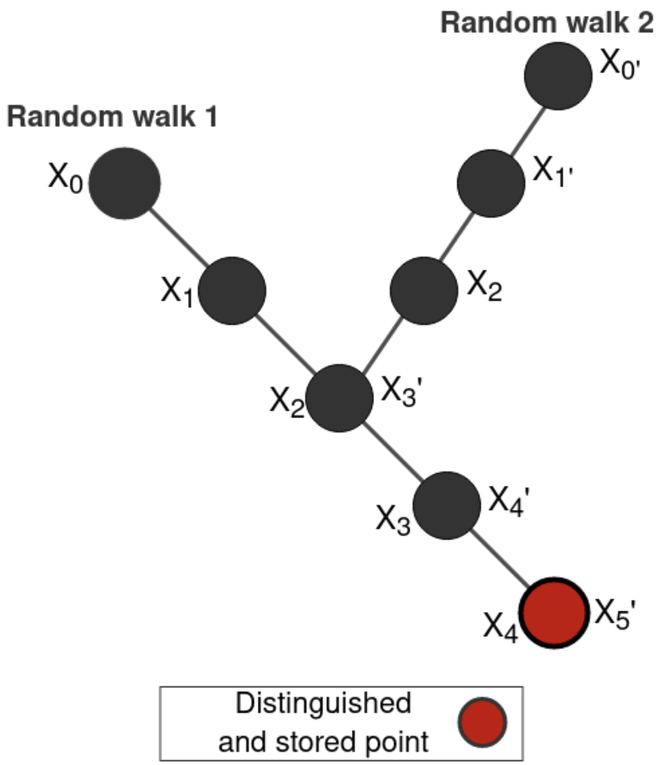

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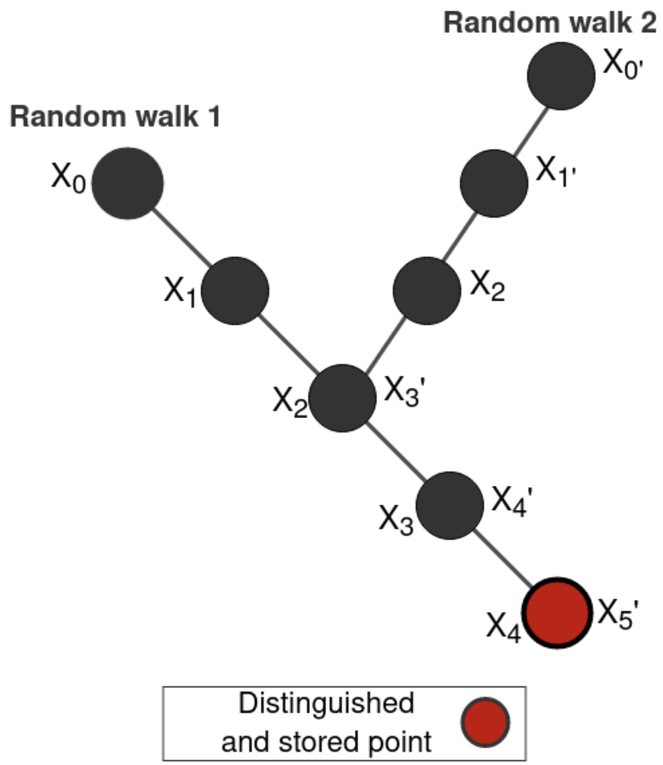

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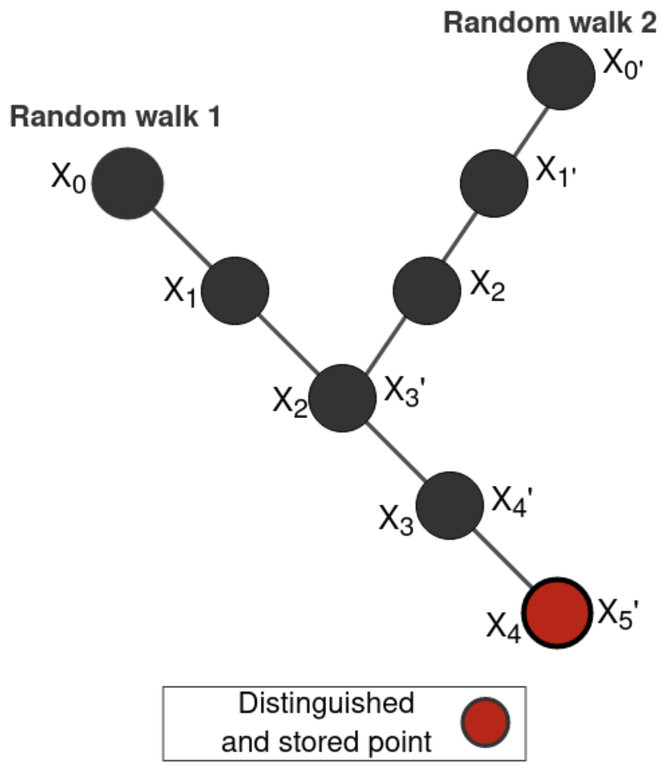

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  ↳ The Birthday paradox

  ↳ (recall) $\sim \sqrt{N}$

Multi-user ECDLP

The multi-user setting

→ We have many users, say L, with the same system parameters.

Example. (ECDLP)

User L_j has $\begin{matrix}\text{public key } Q_j,\\ \text{secret key } x_j\end{matrix}$ such that $Q_j = x_j P$.

Goal. If we can recover the key of one user in $\mathcal{O}($😺$)$,
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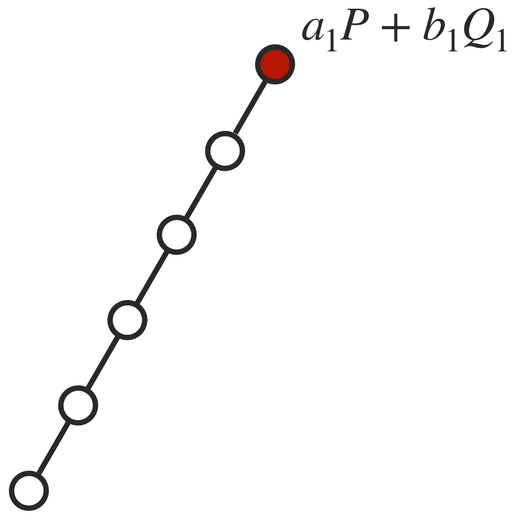

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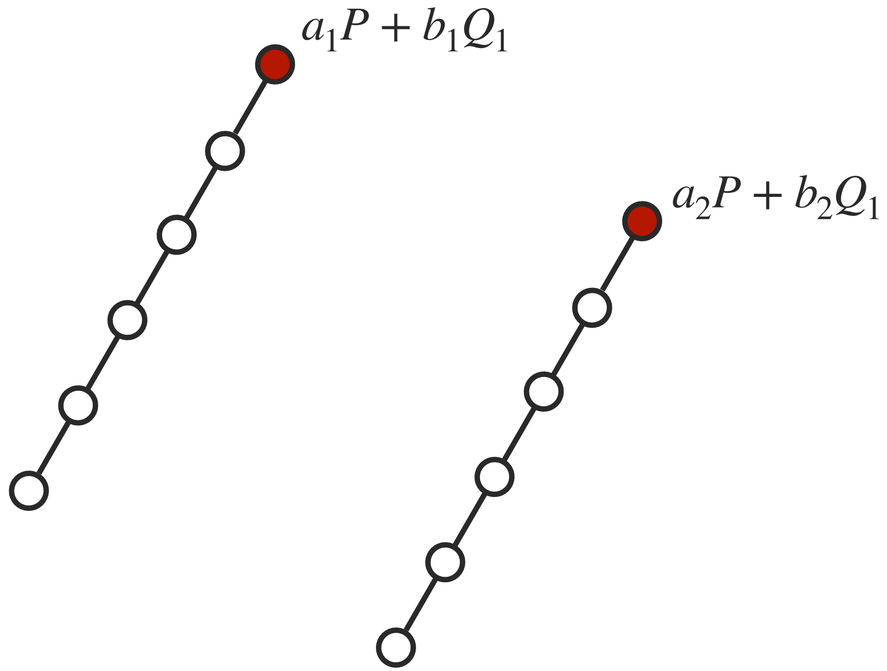

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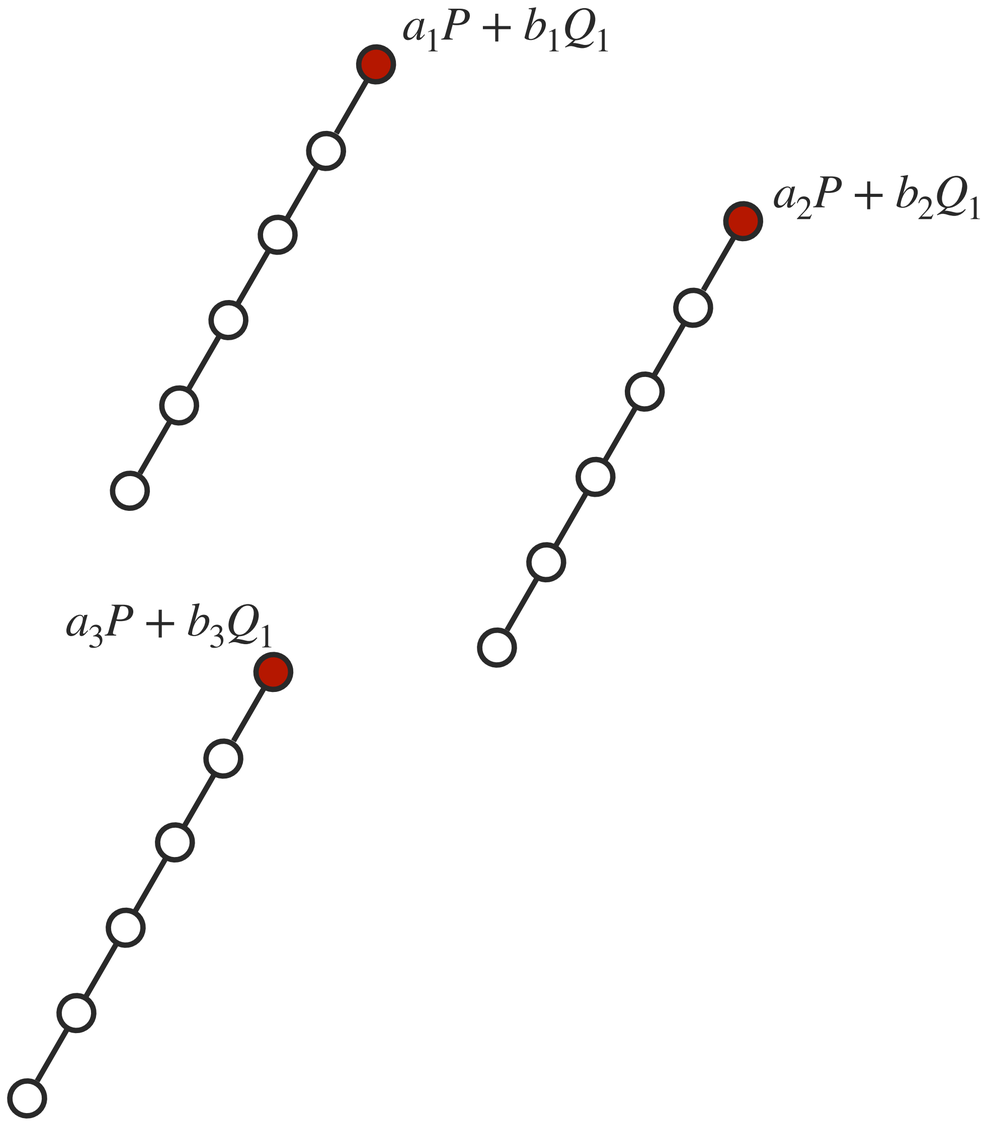

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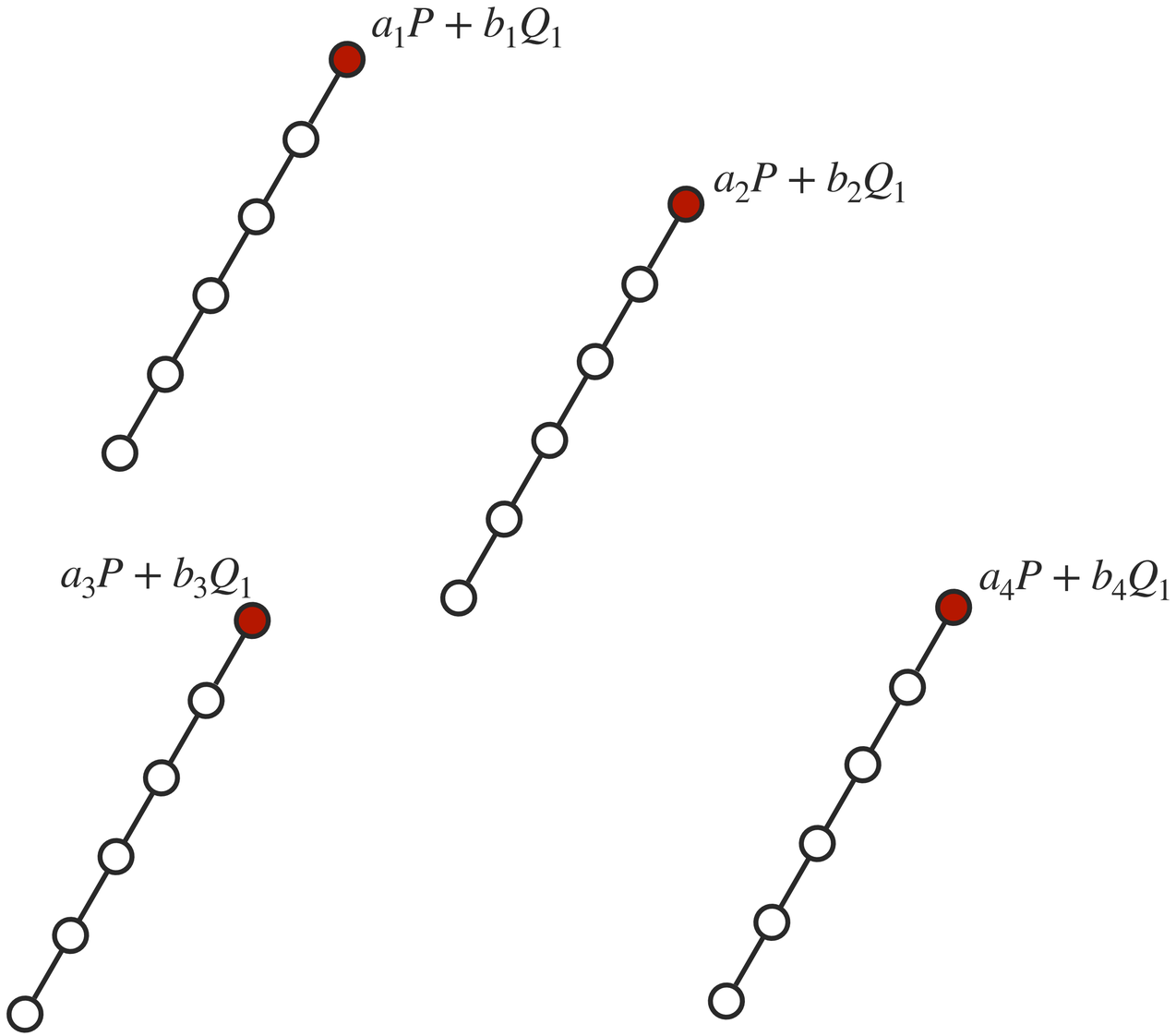

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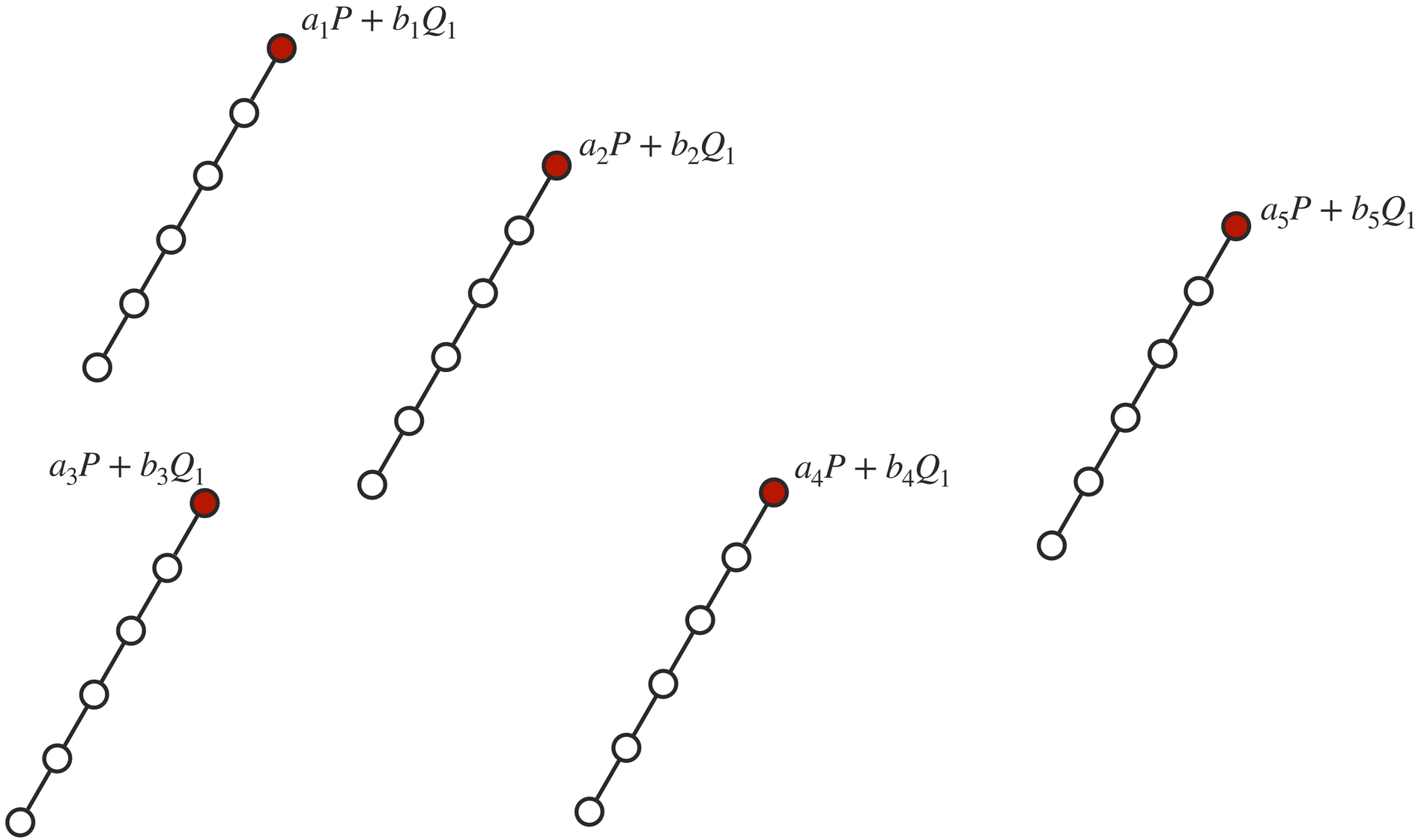

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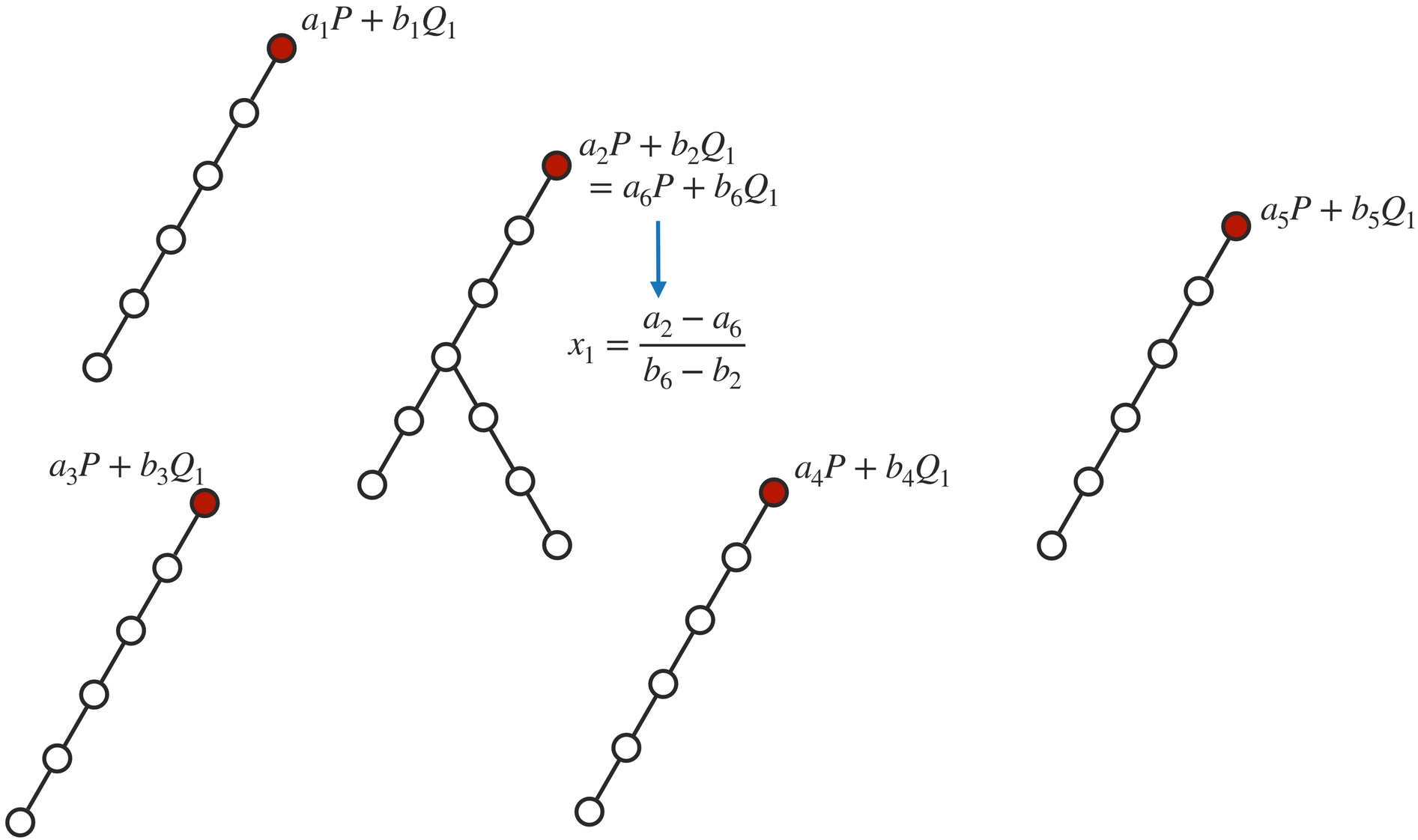

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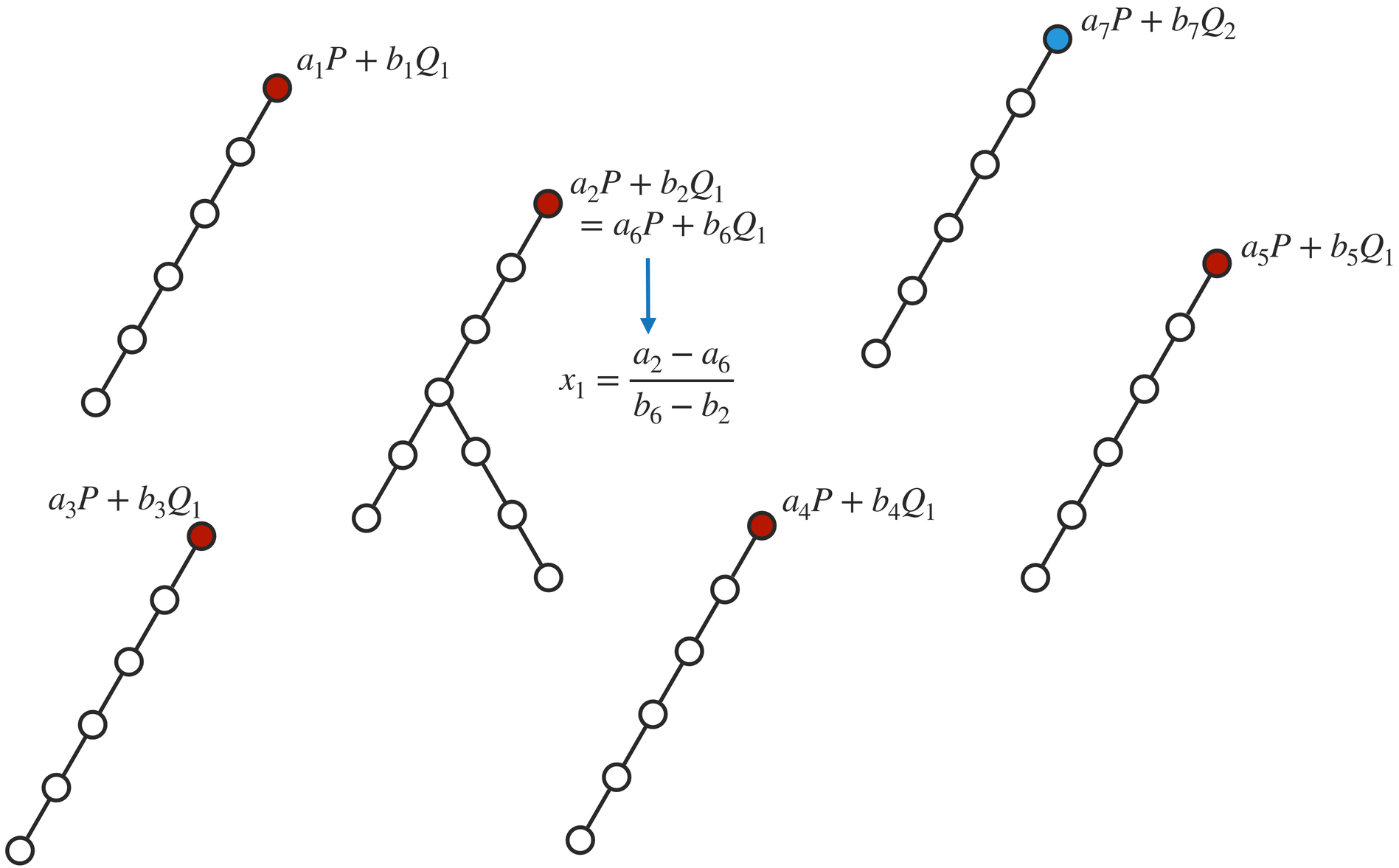

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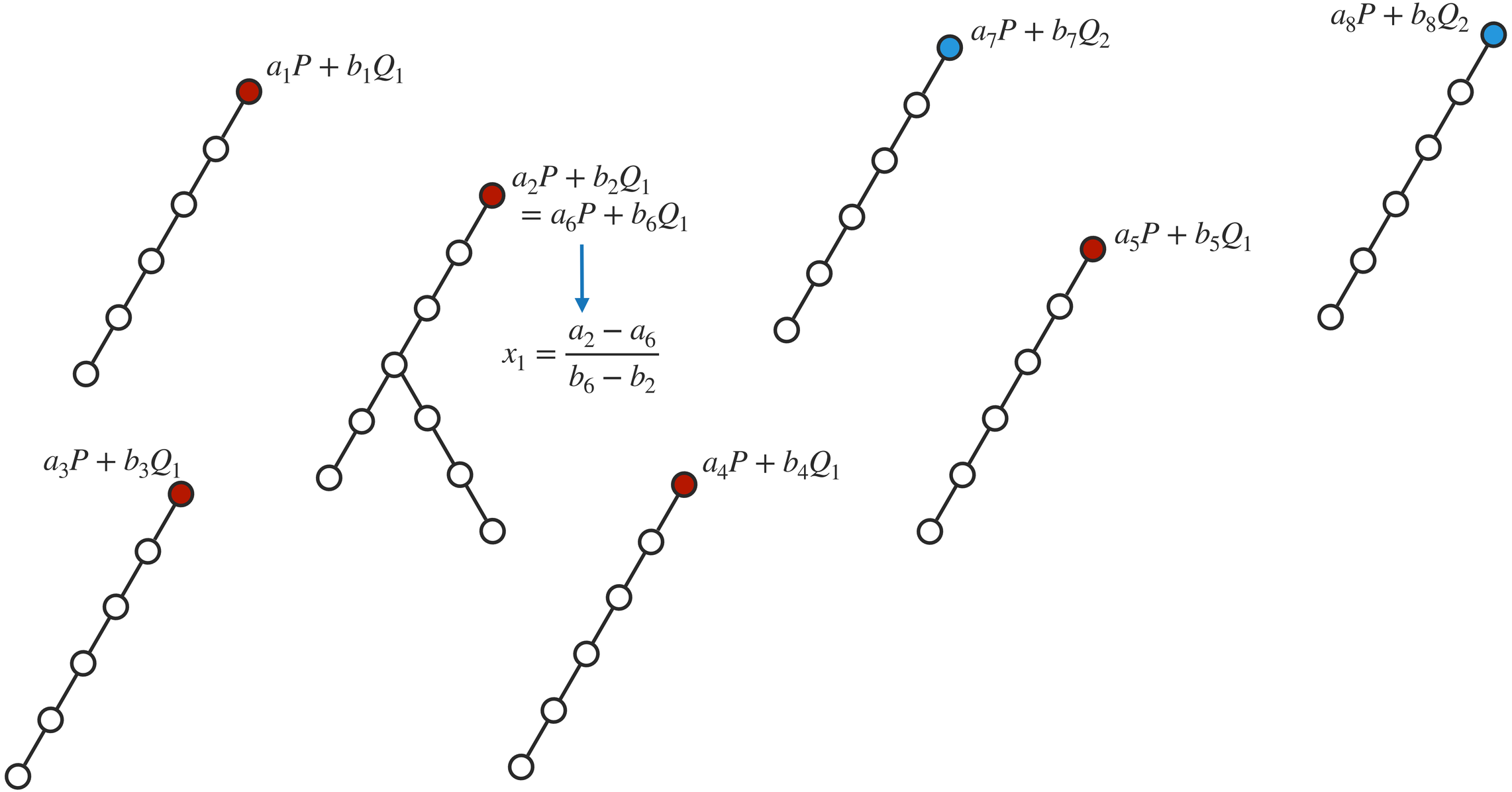

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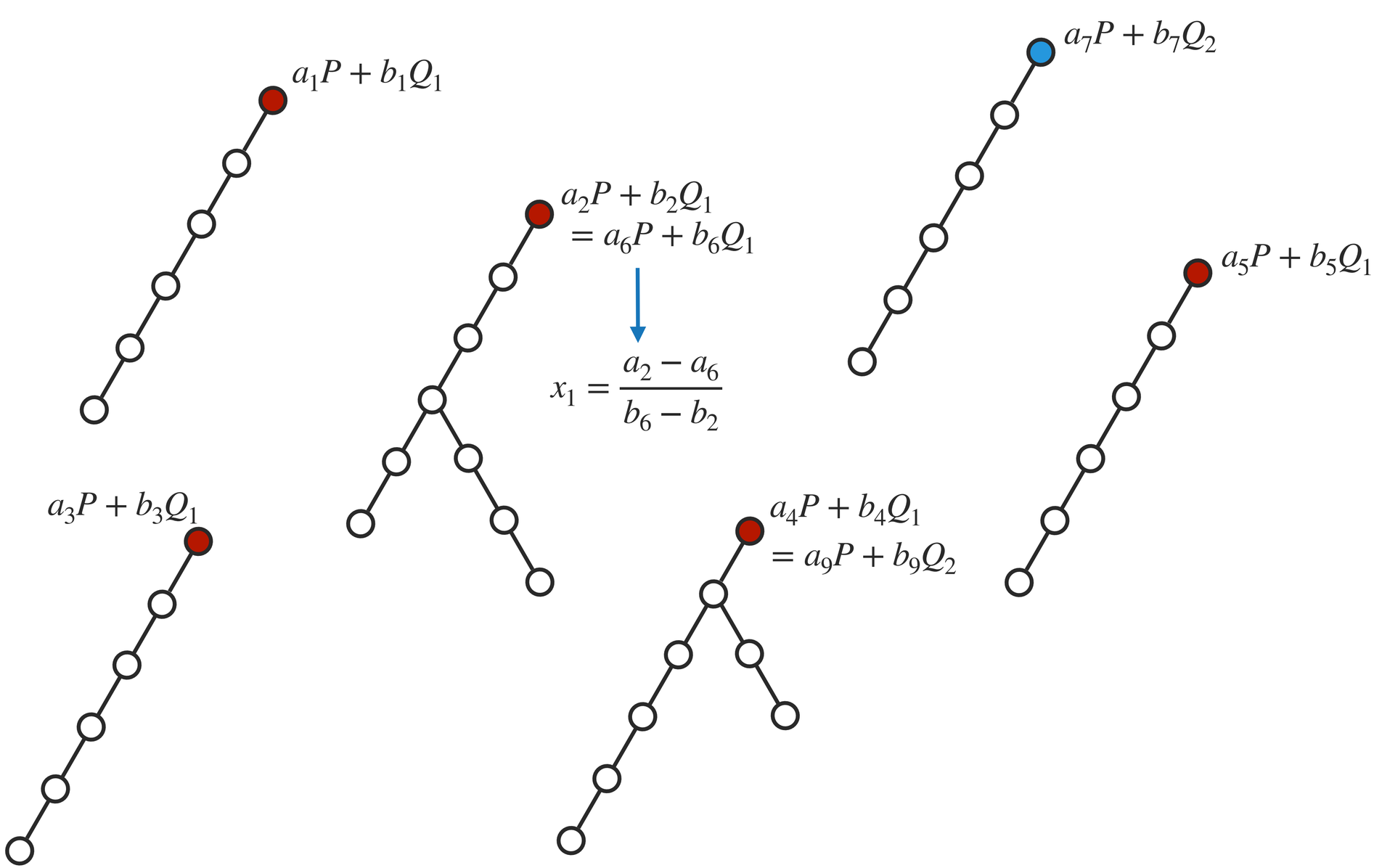

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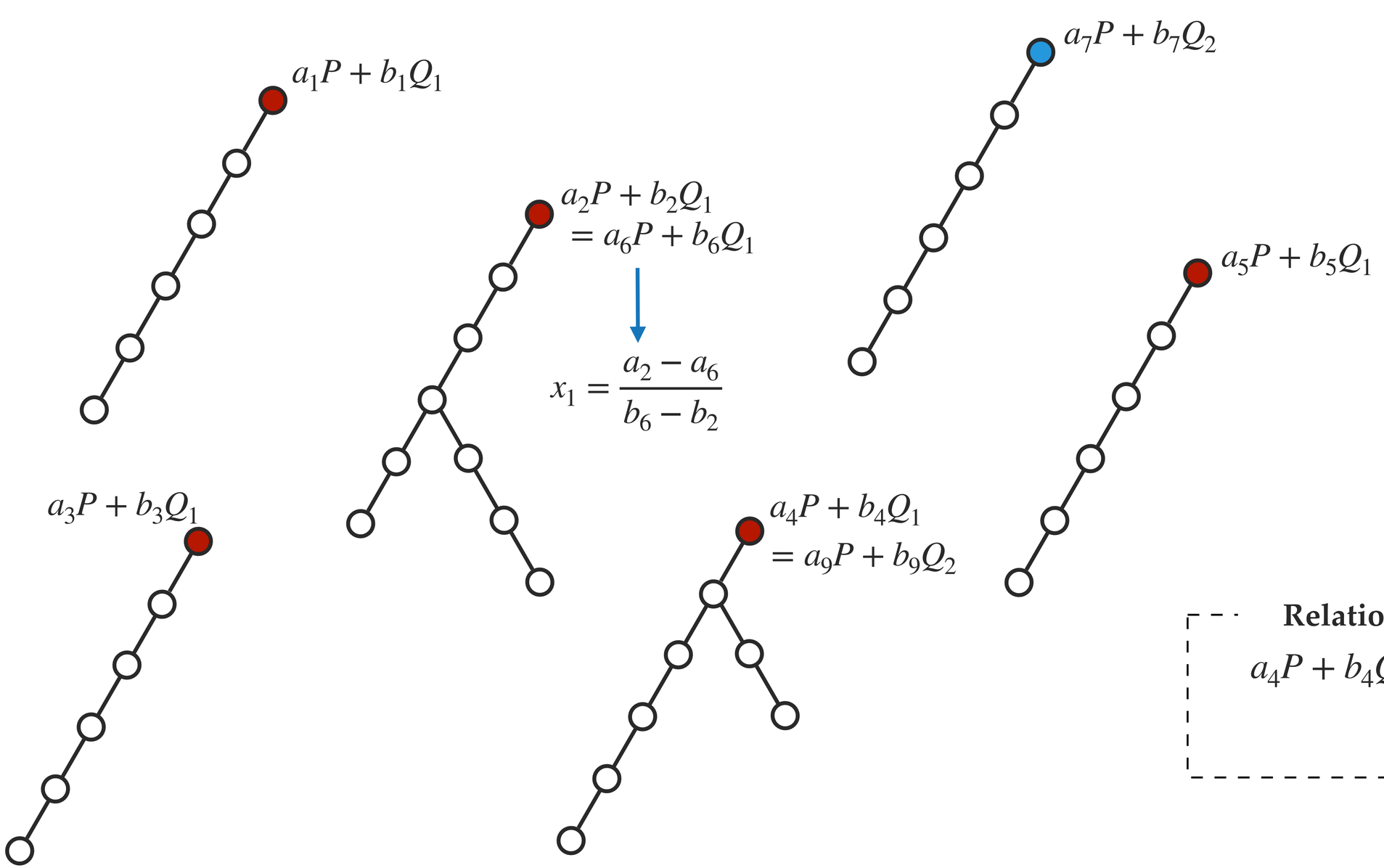

Relation between users

$$a_4P + b_4Q_1 = a_9P + b_9Q_2$$

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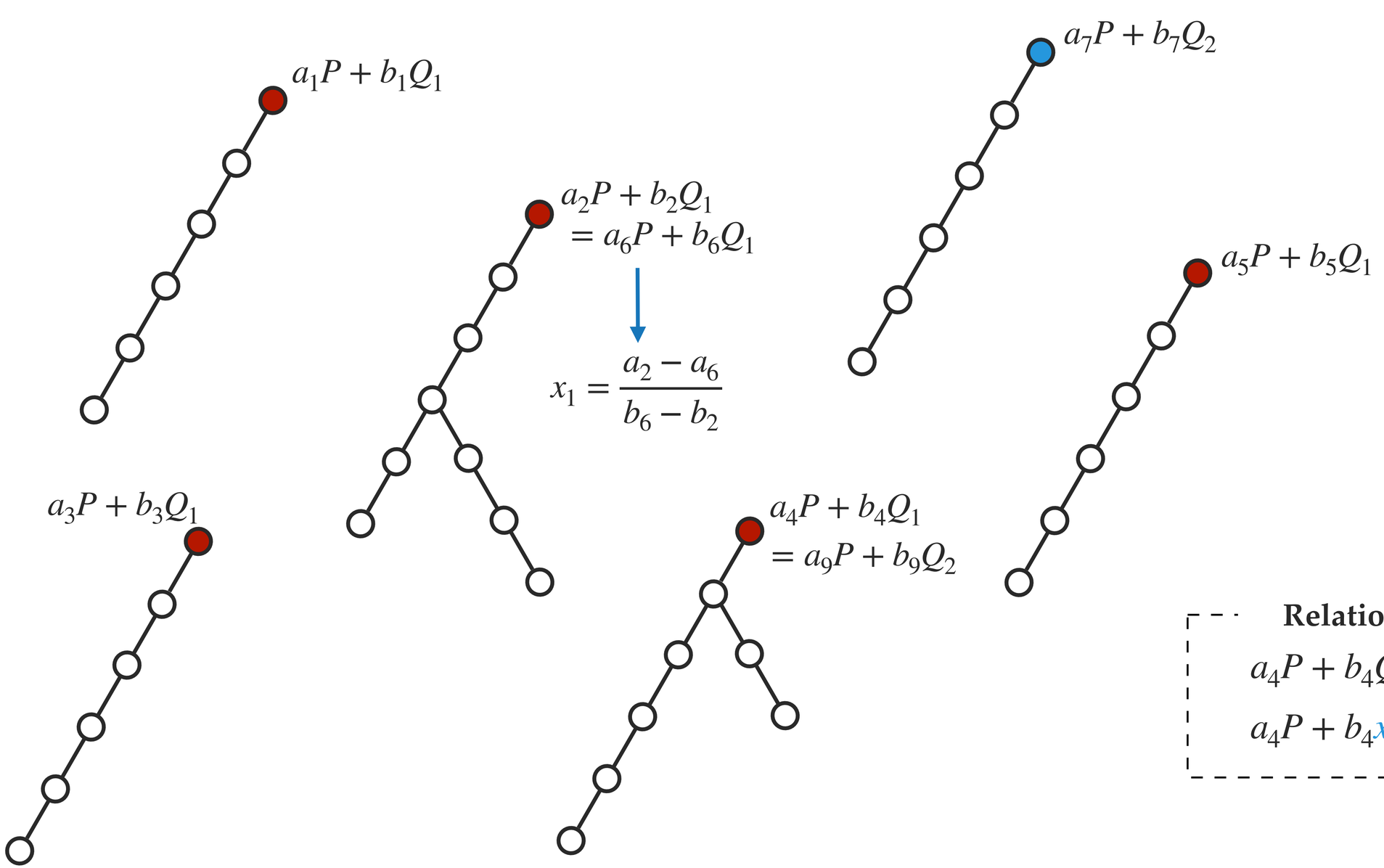

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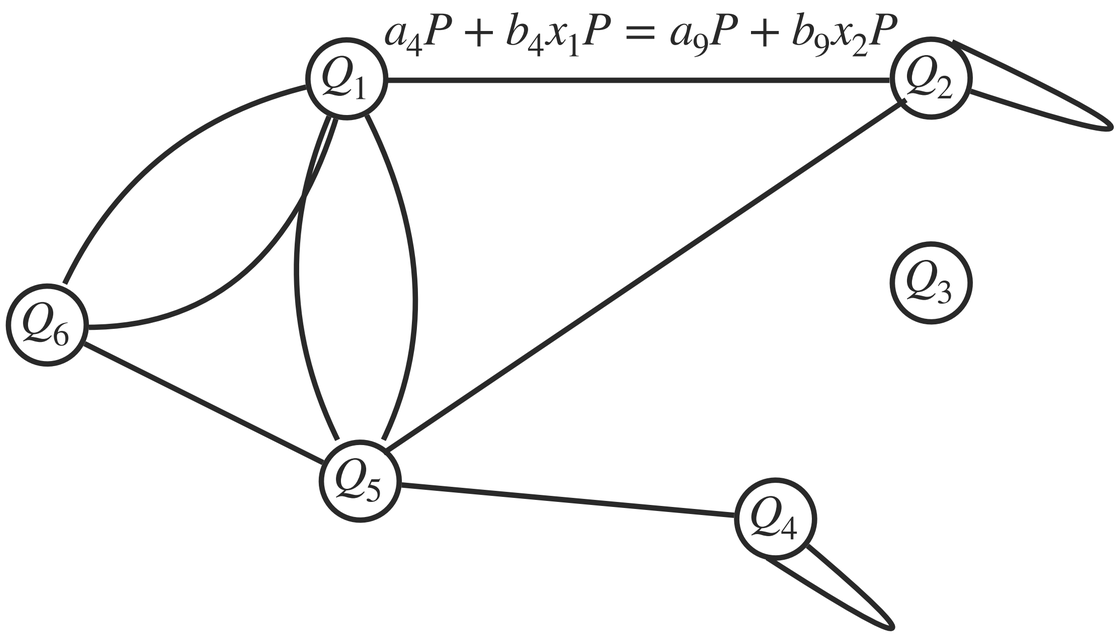
The multi-user graph

(not to be confused with the isogeny graph, seen previously)

Vertices: the public keys of users.

Edges: linear relations between the corresponding secret keys.



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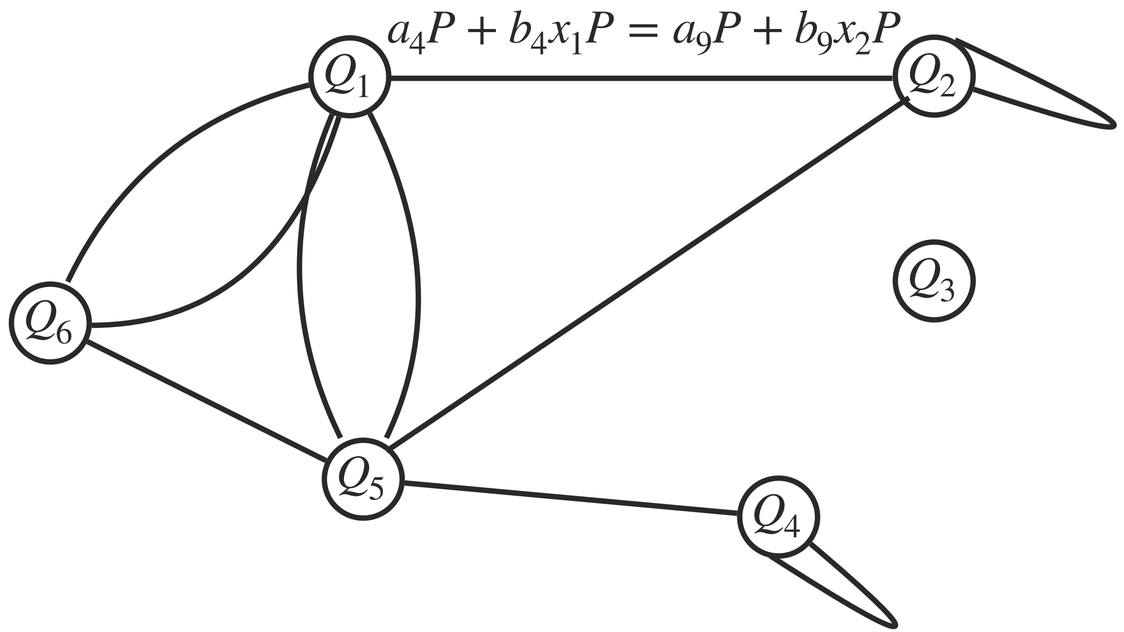

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A user of which we know the secret key

→ Add an infiltrator.

→ Produce cL of these relations.

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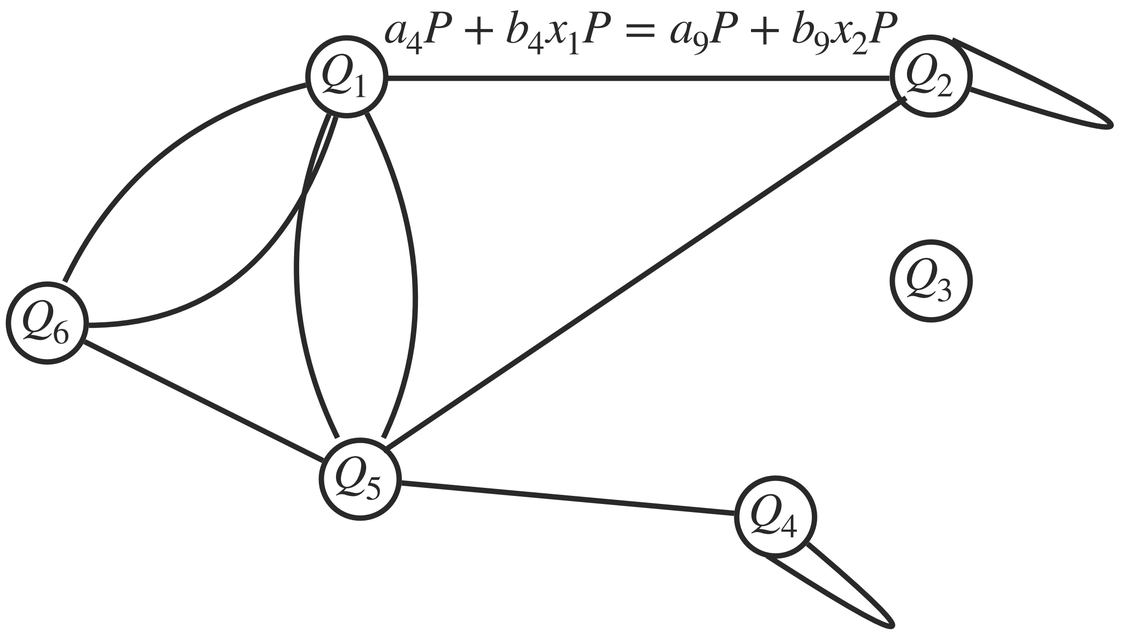

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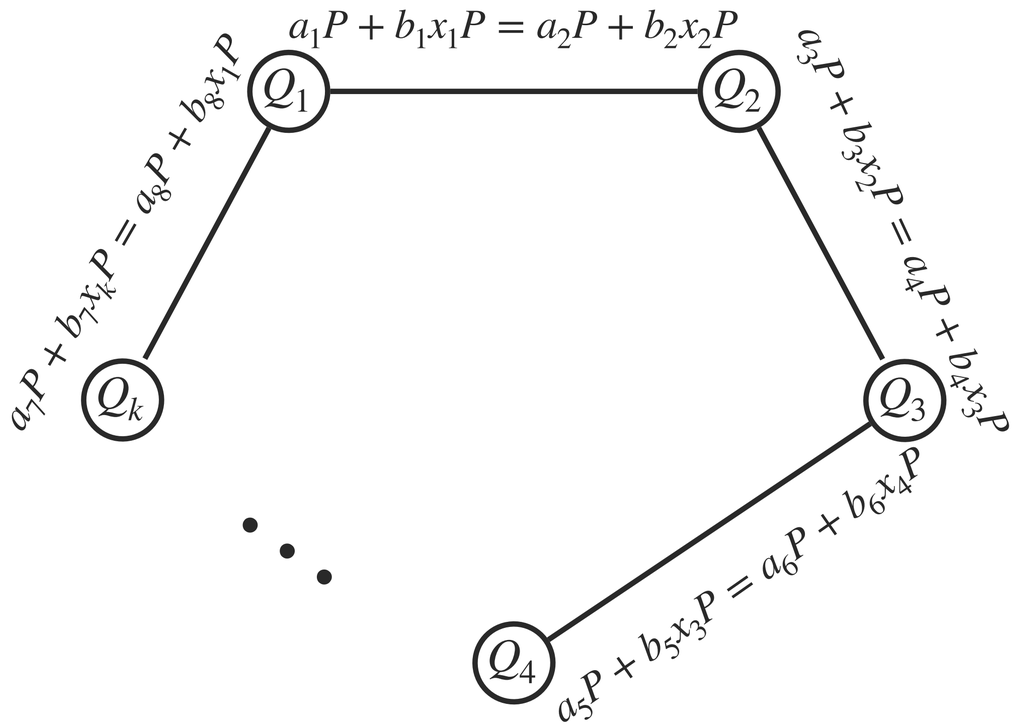
↳ After $2L$ relations, we expect that $0.98L$ vertices are in the largest connected component, including the infiltrator.

Cycles in the multi-user graph



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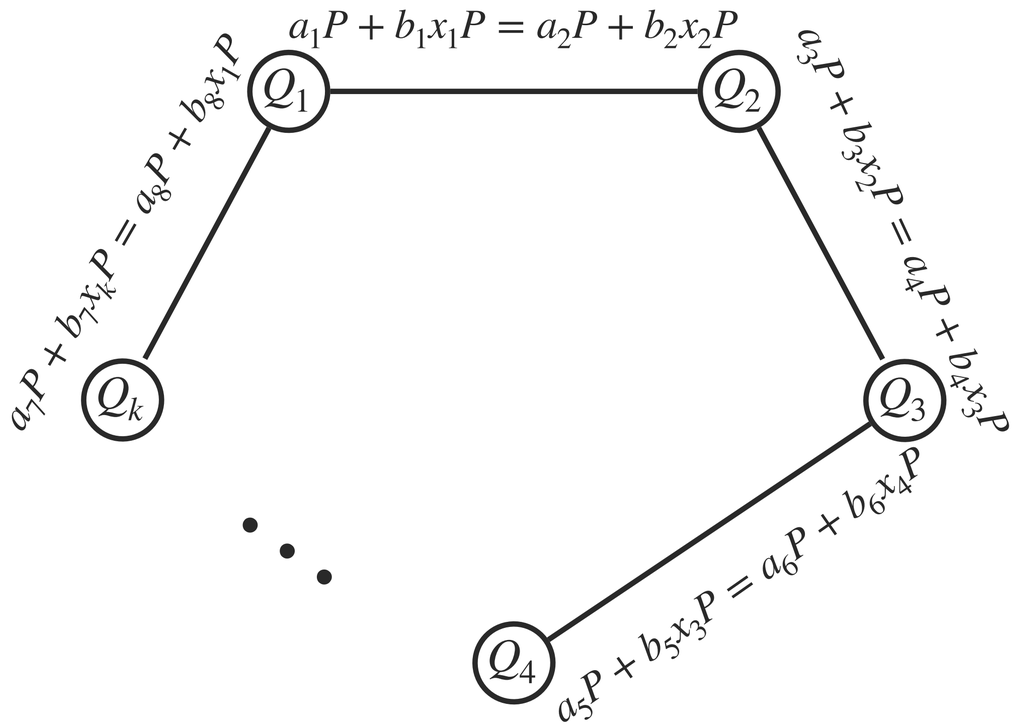

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A cycle of length k yields a linear system of k equations in k unknowns.

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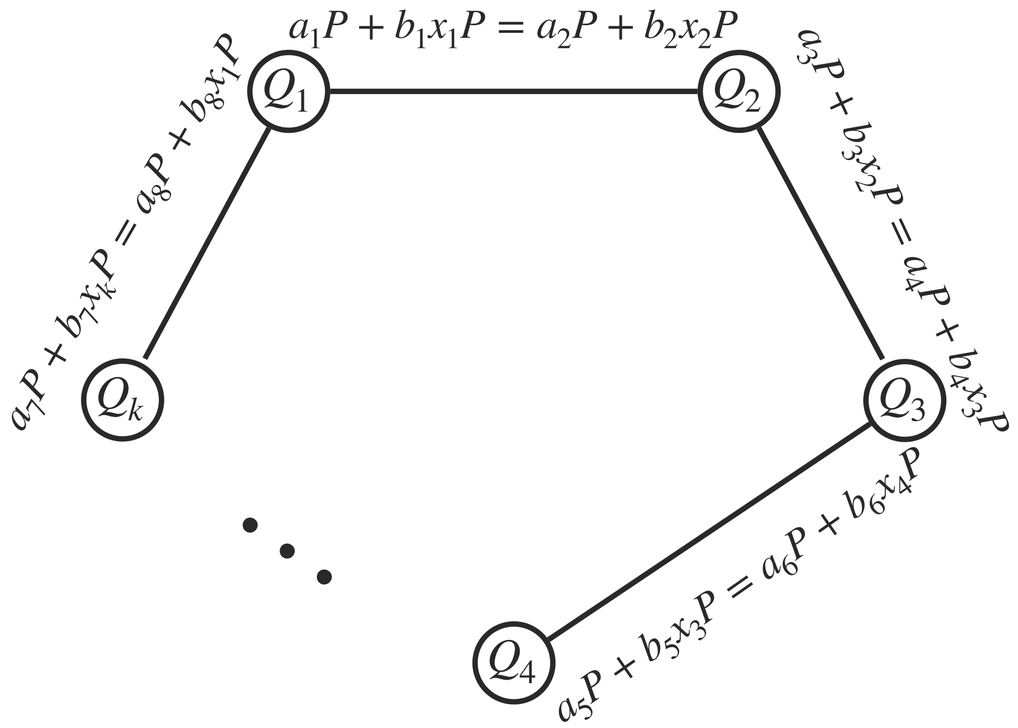

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We solve the linear system and recover the secret key of all users in the cycle.

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≠ as many collisions as users

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Let G be a directed graph where each vertex has exactly one outgoing edge. Then all connected components of G contain a cycle.

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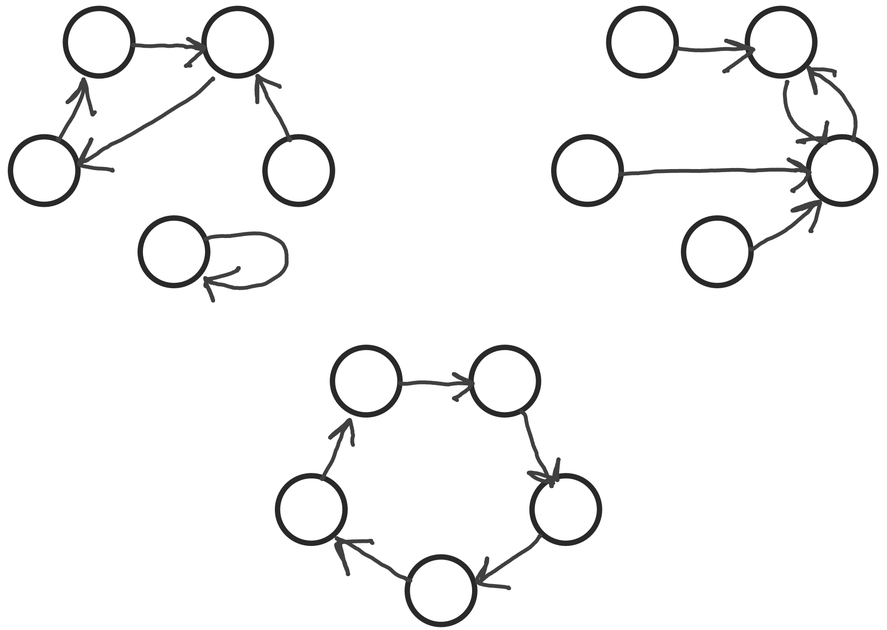

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collision was discovered by L_i.

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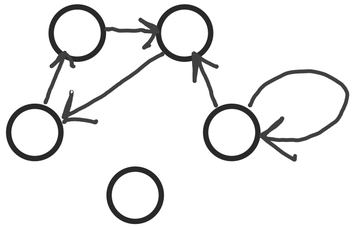

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→ We have a user-independent walk.

→ We can use cycles to recover keys.

→ We can avoid useless collisions.

No need for an infiltrator.

We have an exact count on the number of collisions.

We recover all secret keys.

Fixed-degree isogeny pathfinding

A history lesson

The problem underlying SIDH

! We have $p = c2^e3^f - 1$, for small c

SSI-T (simplified)

Input: Two supersingular elliptic curves E_0 and E_1 defined over $\mathbb{F}_{p^2}$, the restriction of a 2^e-isogeny φ to the 3^f-torsion of E_0.
Question: Find the 2^e-isogeny φ such that $\varphi : E_0 \to E_1$.

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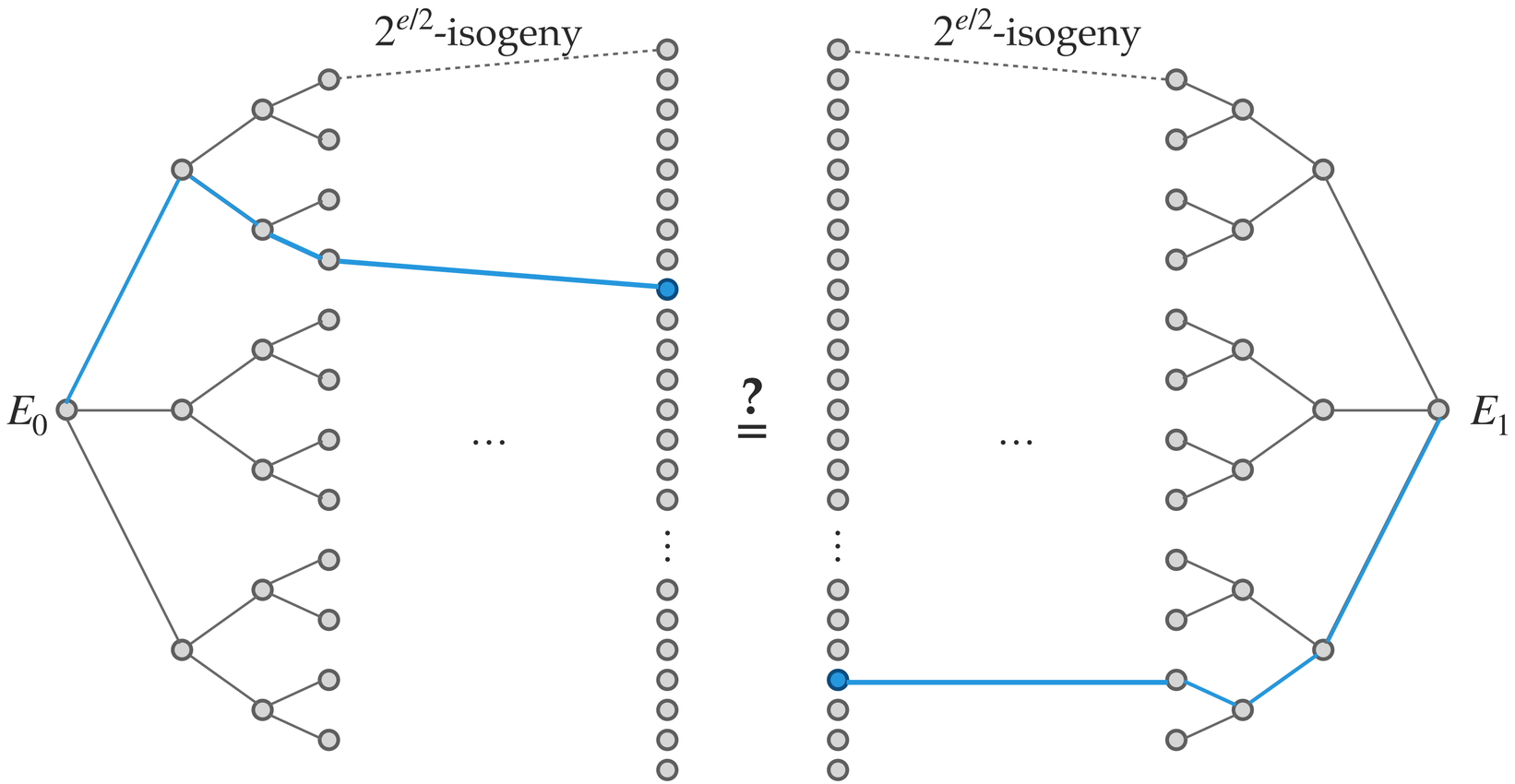
→ The vOW approach has no way of using torsion information.

↳ We look at the problem simply as a fixed-degree isogeny pathfinding problem.

Meet-in-the-middle

Goal: Find a 2^e-isogeny from E_0 to E_1.



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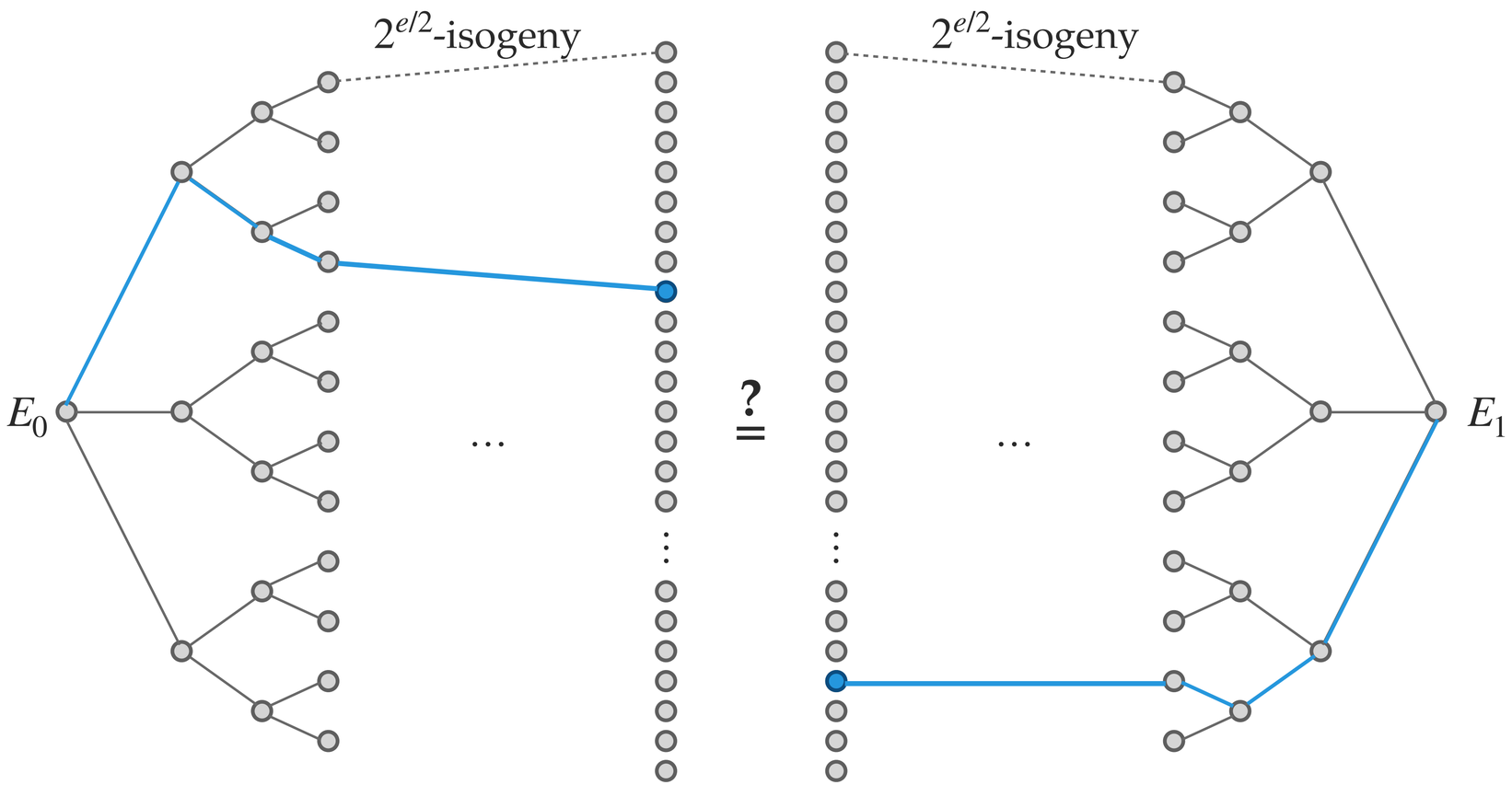

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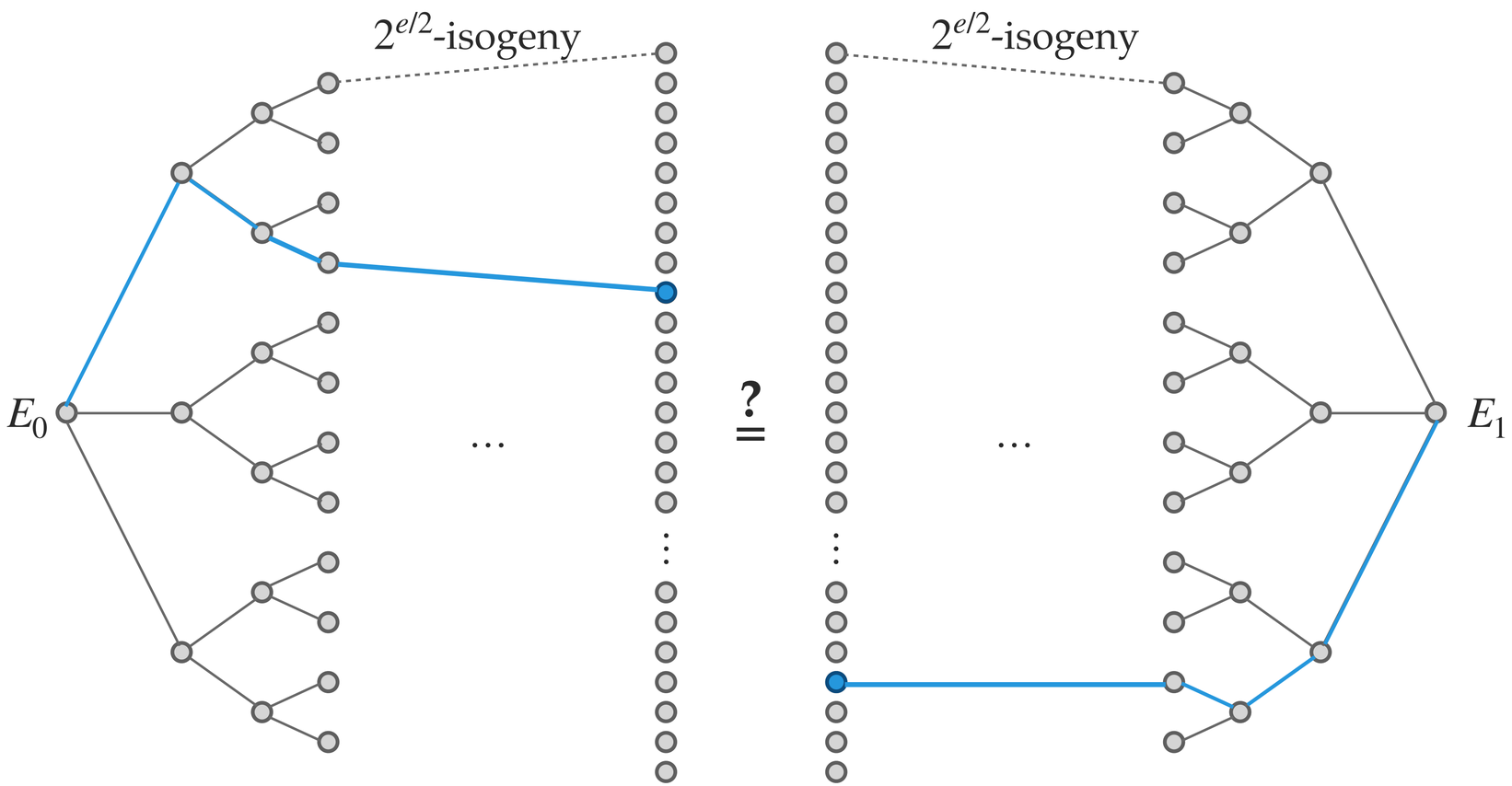

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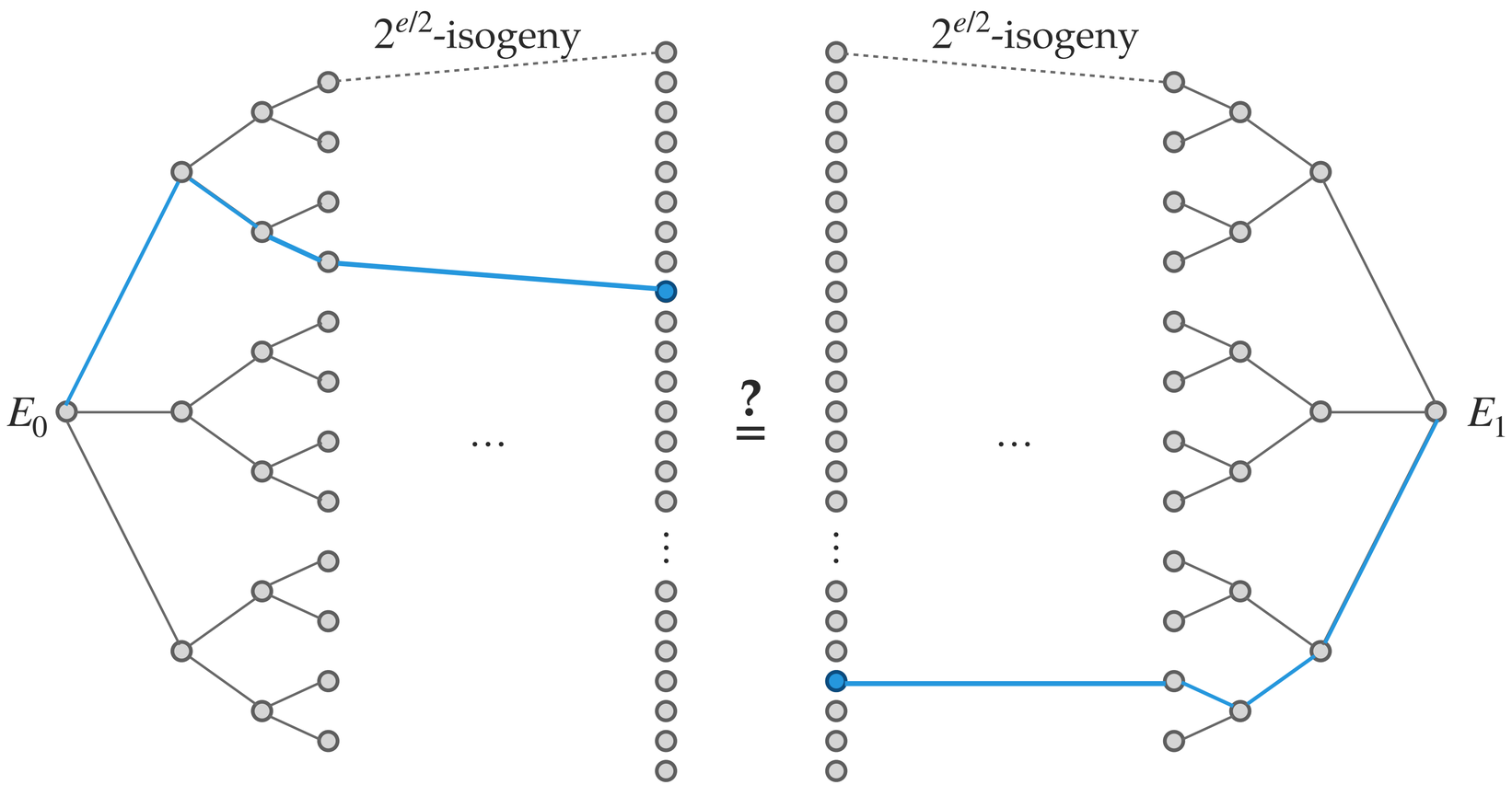

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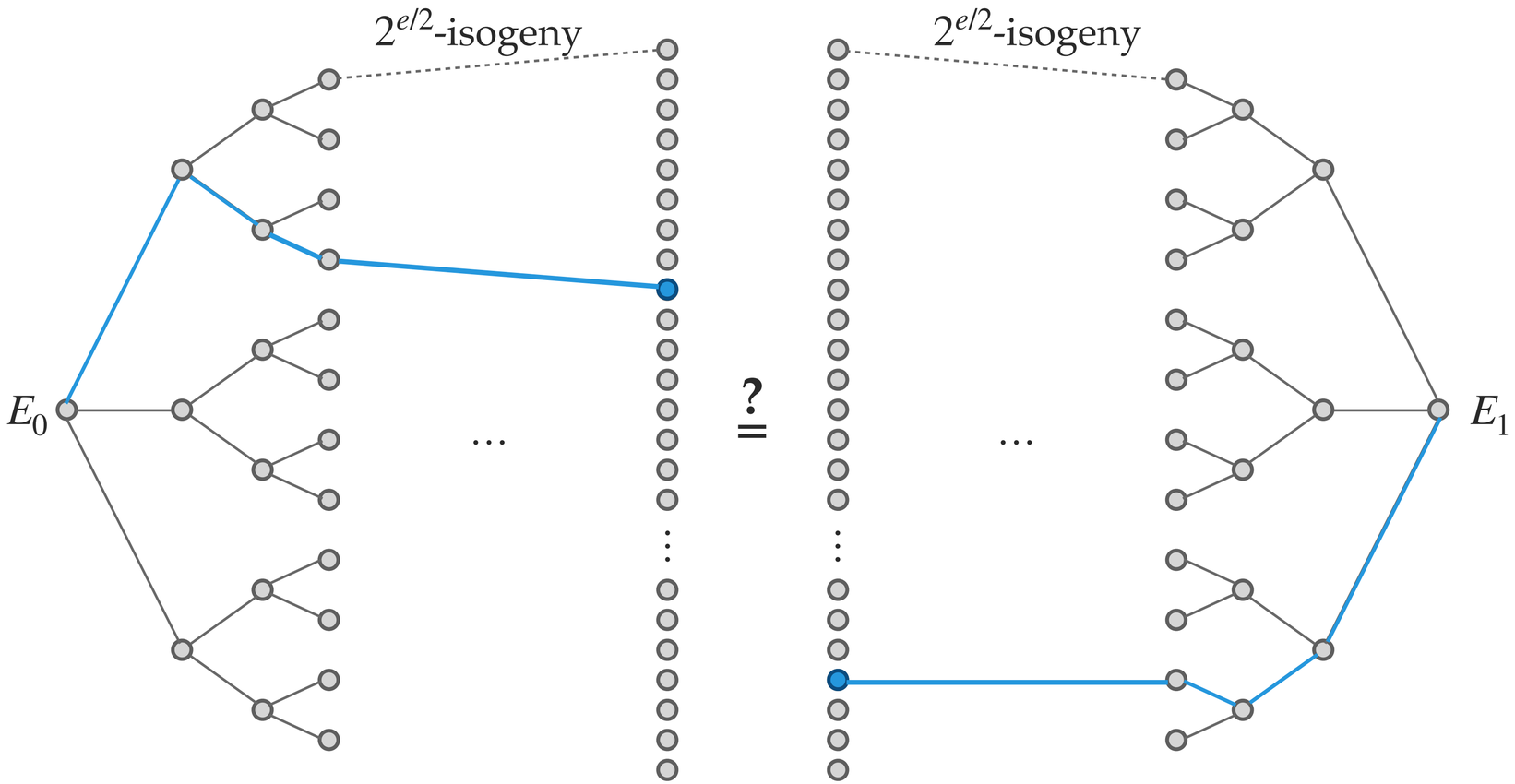
$\mathcal{O}(2^{e/2})$

in both time and memory.

*Trade-offs are possible.

vOW for fixed-degree isogeny pathfinding



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$I = \{1,\ldots,2^{e/2}\}$ - indices of possible paths
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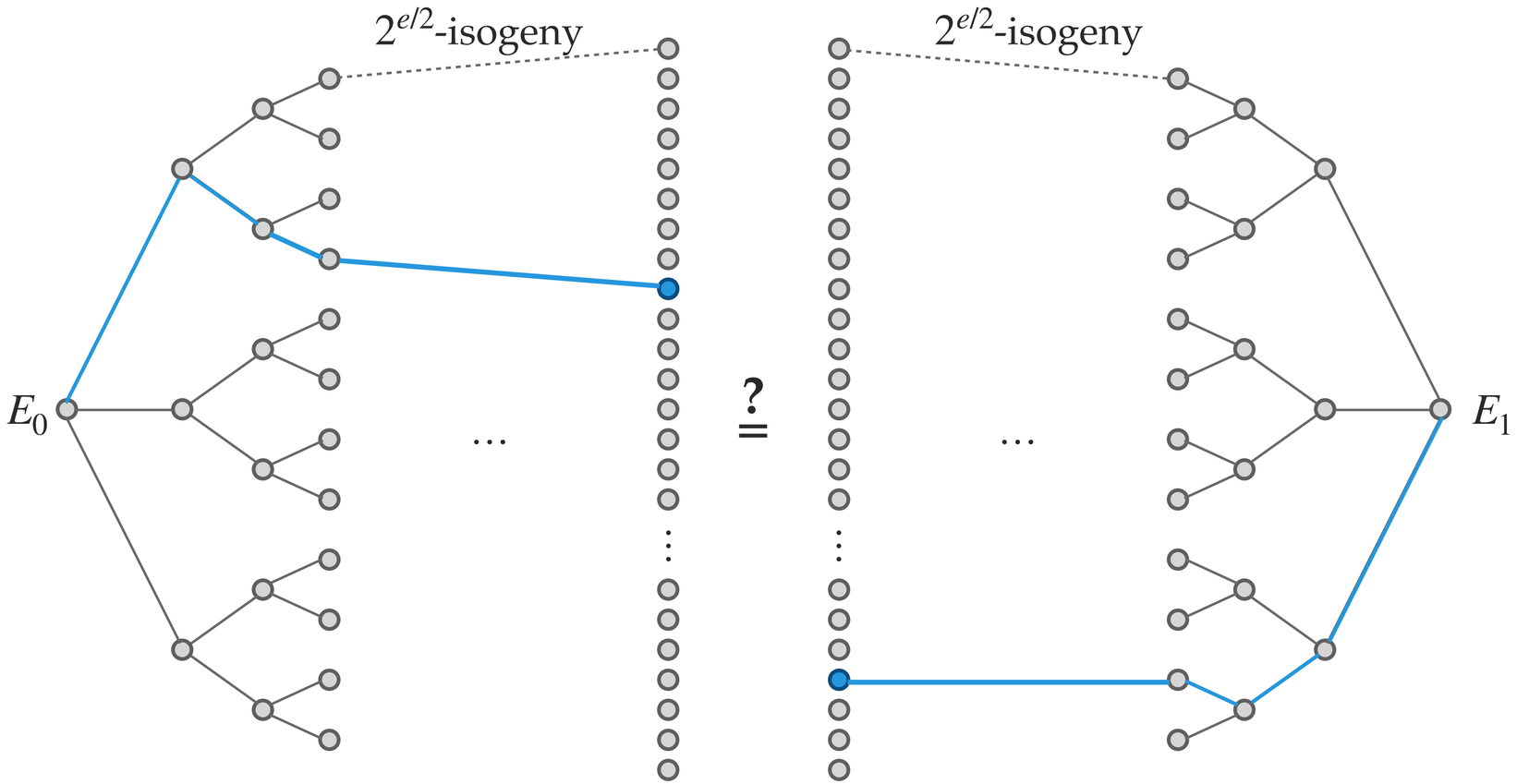

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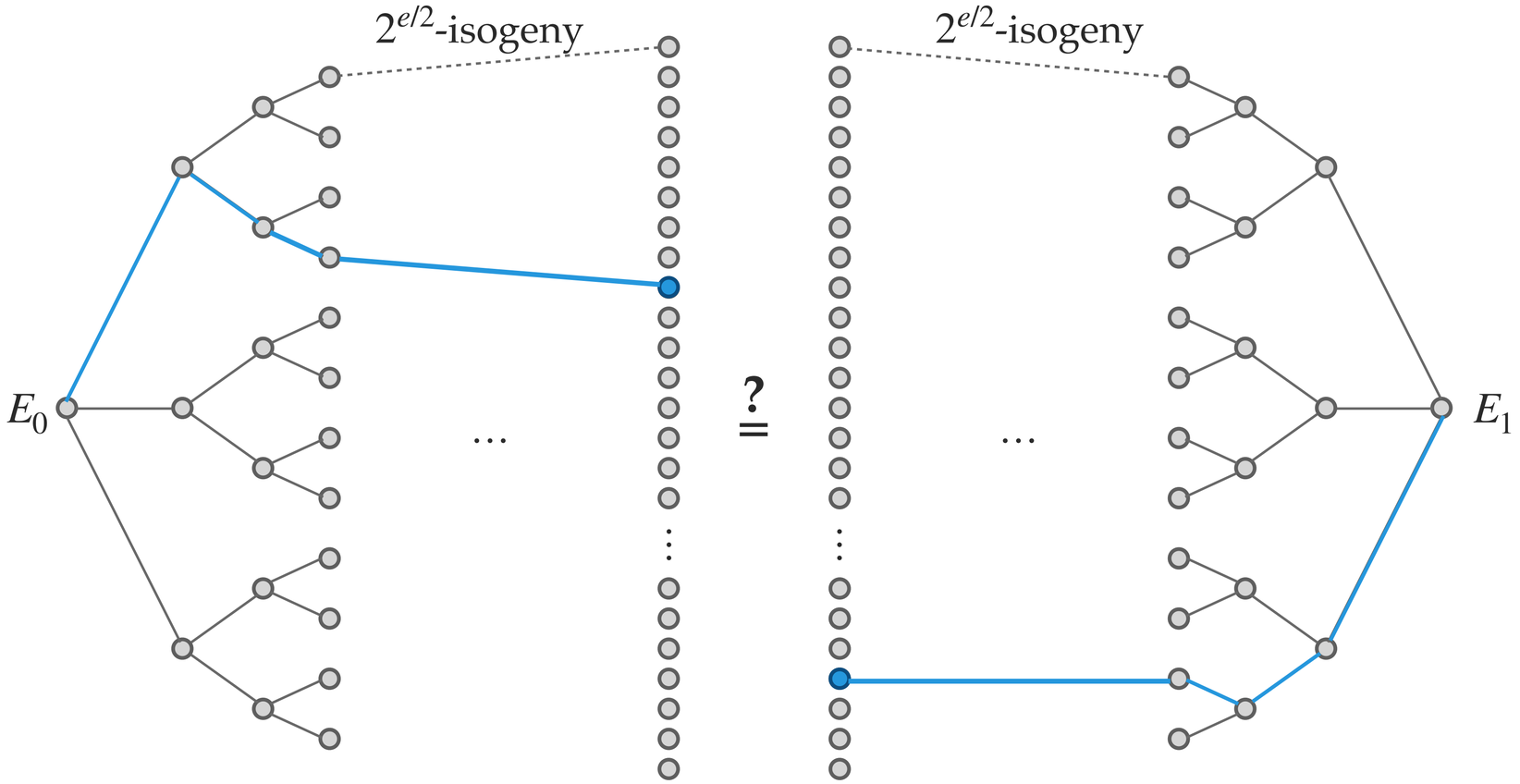
Taking the j-invariant



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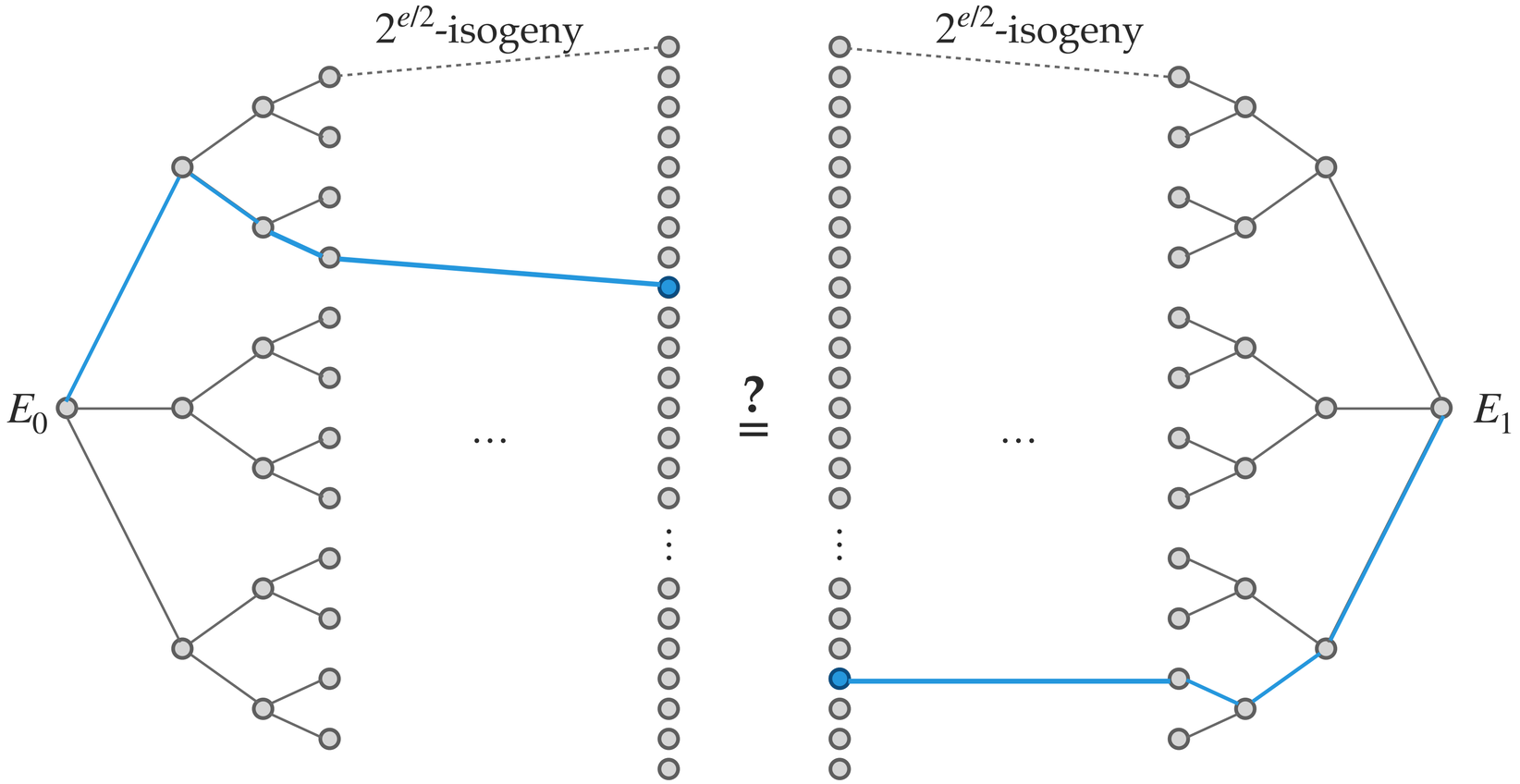

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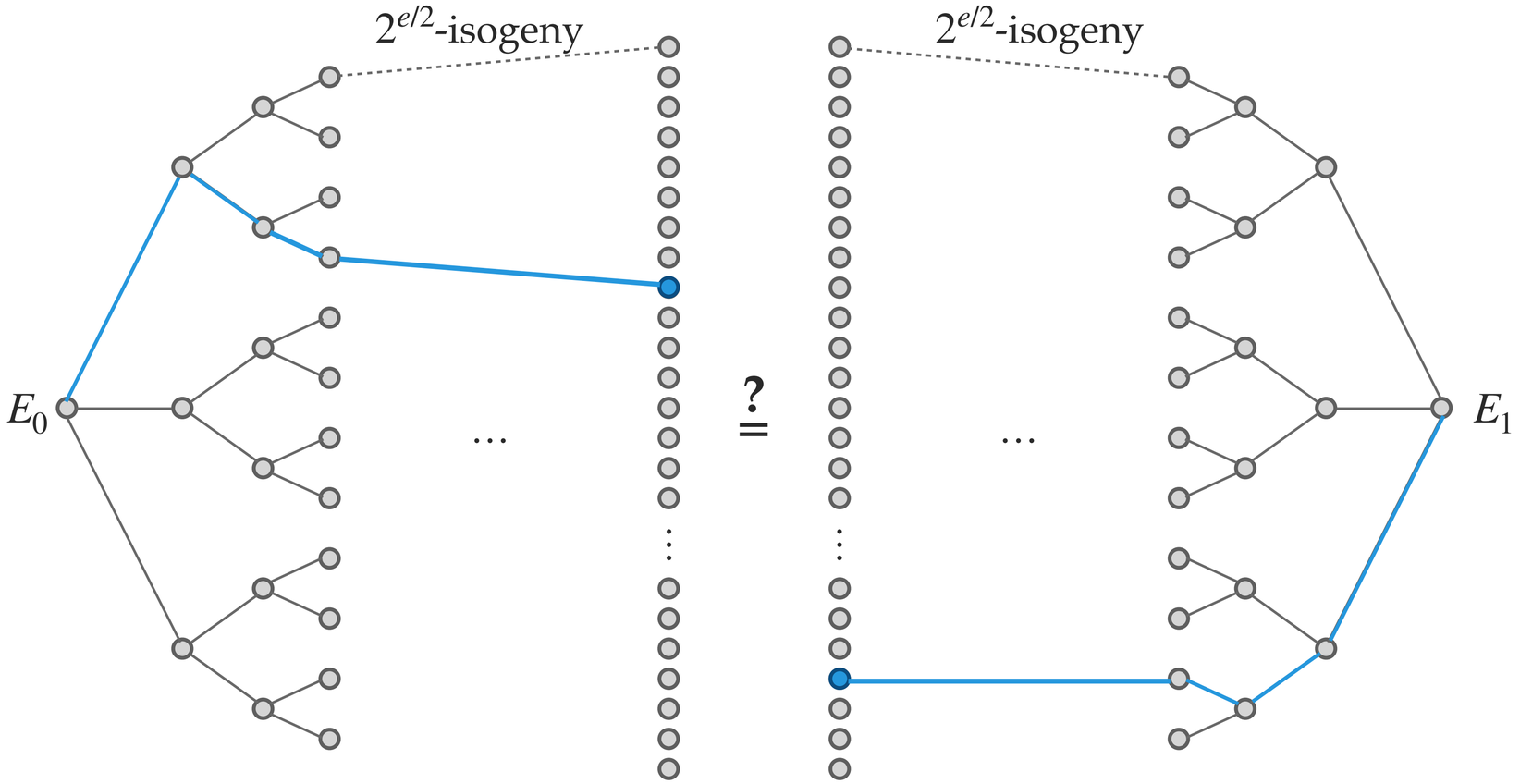

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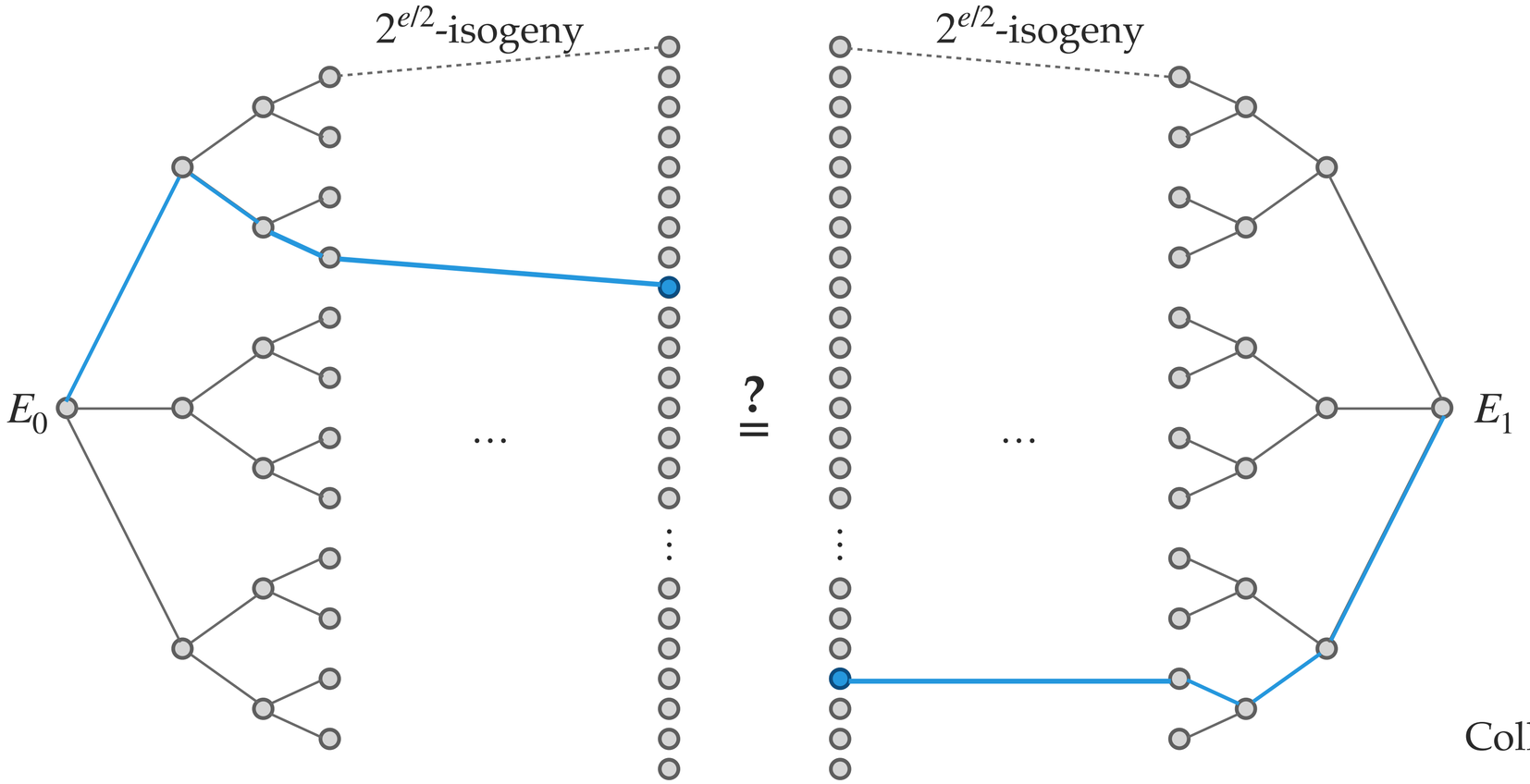

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Collision of H

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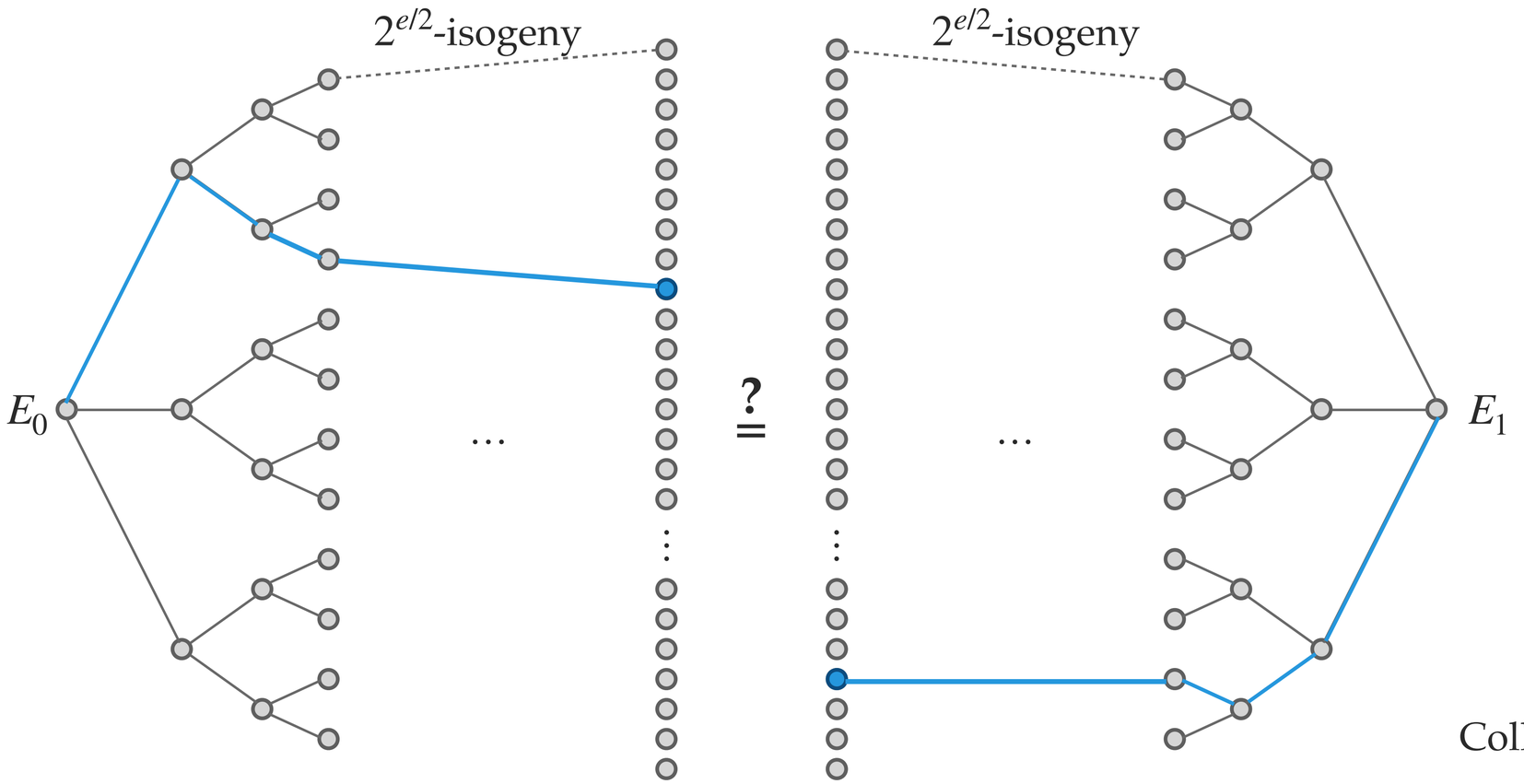

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Collision of H
↗ collision of F : golden
↘ collision of g : a hash collision

A different view of vOW for ECDLP

vOW walk for ECDLP

$$f(R) = \begin{cases} R + c_1 P & \text{if } R \in S_1 \\ R + c_2 P & \text{if } R \in S_2 \\ \cdots & \\ R + c_k P & \text{if } R \in S_k, \end{cases}$$

recall: $S = \{R \in E(\mathbb{F}_q)\}$

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↳ Every collision of H is a collision of F. 😺

To sum up

g is injective ⟶ no collisions

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We did this by using the current element of S, not just to decide where to walk next, but also in the computation of the next step.

→ g is injective → no collisions

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We were able to do this because we are able to walk between elements of S while making sure we do not step out of S.

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We are able to do this simply because we are using the entire domain

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(when $p = c\ell^e - 1$, for some small prime ℓ and a small factor c)

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- $f(j_i) = \Phi_\ell(j_{i-1}, j_i, j_i \mod \ell)$
  - The modular polynomial
  - The previous j-invariant, so that we can avoid backtracking
  - The choice of a root

Why $\sqrt{}$?

- The Meet-in-the-middle algorithm
- The van Oorschot and Wiener algorithm
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↳ because of the ratio between $\mathbb{F}_p$-rational curves and all curves.

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The sta

The vC

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To sum up

→ We have a user-independent walk.

→ We can use cycles to recover keys.

→ We can avoid useless collisions.

No need for an infiltrator.

We have an exact count on the number of collisions.

We recover all secret keys.

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User-independent walk ✓

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Taking the Montgomery coefficient

The ℓ_n-order subgroup

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Multi-user isogeny pathfinding

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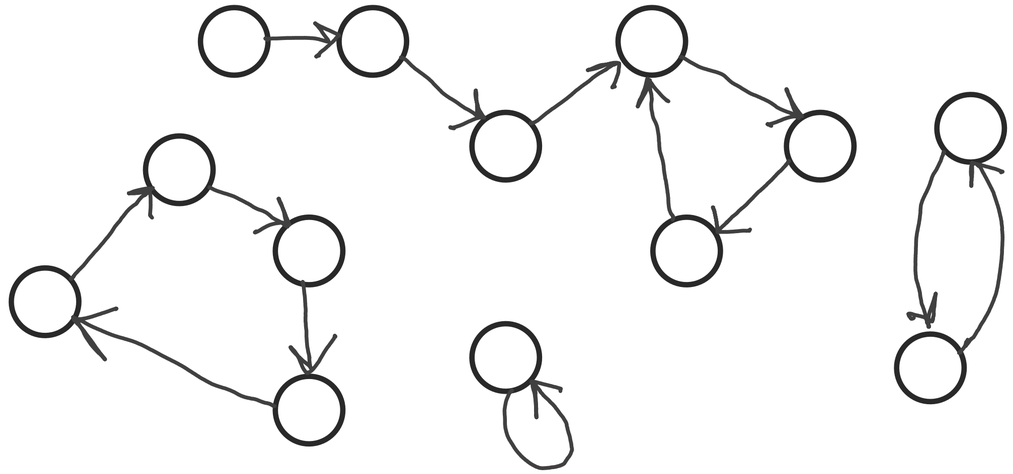

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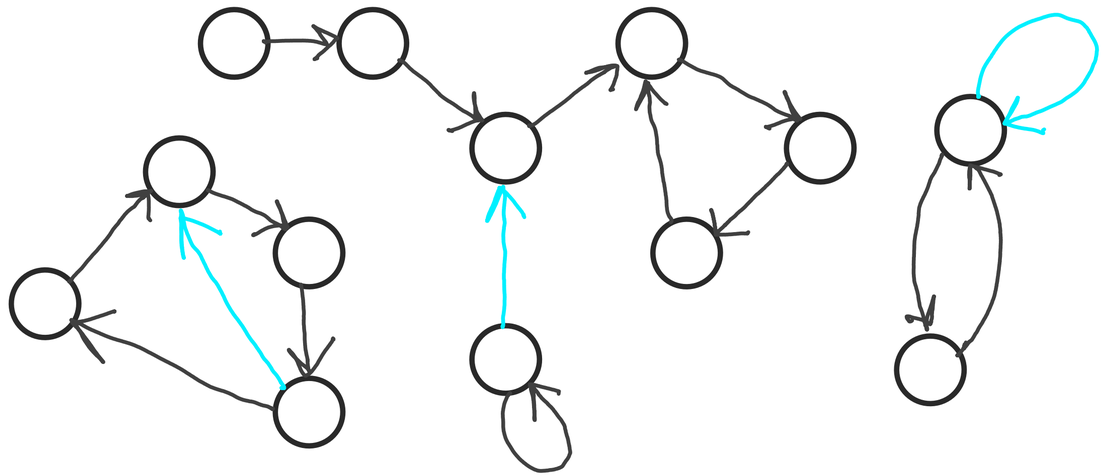

→ Now, we add one more collision per component.

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A cycle is yet another $\mathbb{F}_p$-rational endomorphism.

From nb. of collisions to complexity

Theorem. Let S be a finite set of cardinality N, let $f : S \to S$ be an efficiently-computable semi-random function. The expected number of steps of a random walk using the f mapping before k collisions occur is asymptotic to

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Thank you !

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