Radboud University

# Digital Signatures from Matrix Code Equivalence

**Monika Trimoska** (based on joint work with **Ruben Niederhagen, Edoardo Persichetti, Tovohery Hajatiana Randrianarisoa, Krijn Reijnders** and **Simona Samardjiska**)

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# Matrix Code Equivalence (MCE)

# The Matrix Code Equivalence Problem

**Matrix code $\mathcal{C}$:** a subspace of $\mathcal{M}_{m\times n}(\mathbb{F}_q)$ of dimension $k$ endowed with **rank metric**

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**Input:** Two $k$-dimensional matrix codes $\mathcal{C},\mathcal{D}\subset\mathcal{M}_{m\times n}(\mathbb{F}_q)$
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**Known:** Any isometry $\mu:\mathcal{C}\to\mathcal{D}$ can be written, for some $\mathbf{A}\in\mathrm{GL}_m(q),\mathbf{B}\in\mathrm{GL}_n(q)$, as

$$\mathbf{C}\mapsto\mathbf{ACB}\in\mathcal{D}$$

# Known results [Couvreur, Debris–Alazard & Gaborit, 2021]

$$\mu : \mathbf{C} \mapsto \mathbf{ACB} \in \mathcal{D}, \quad \text{with } \mathbf{A} \in \mathrm{GL}_m(q) \text{ and } \mathbf{B} \in \mathrm{GL}_n(q)$$

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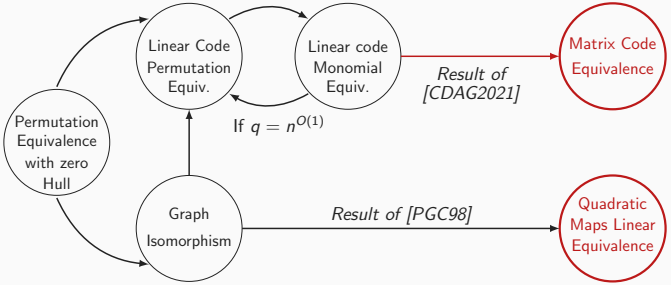

# What is QMLE?

# Multivariate crypto basics

- systems of multivariate polynomials $\mathcal{P} = (p_1, p_2, \ldots, p_k)$, every $p_s$ polynomial in $N$ variables $x_1, \ldots, x_N$

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$$p_s(x_1, \ldots, x_N) = \sum \gamma_{ij}^{(s)} x_i x_j + \sum \beta_i^{(s)} x_i + \alpha^{(s)}, \qquad \alpha^{(s)}, \beta_i^{(s)}, \gamma_{ij}^{(s)} \in \mathbb{F}_q$$

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**Quadratic Maps Linear Equivalence (QMLE) problem**

QMLE$(N, k, \mathcal{F}, \mathcal{P})$:

**Input:** Two $k$-tuples of quadratic maps

$\mathcal{F} = (f_1, f_2, \ldots, f_k)$, $\mathcal{P} = (p_1, p_2, \ldots, p_k) \in \mathbb{F}_q[x_1, \ldots, x_N]^k$

**Question:** Find – if any – $\mathbf{S} \in \mathrm{GL}_N(q), \mathbf{T} \in \mathrm{GL}_k(q)$ such that

$$\mathcal{P}(\mathbf{x}) = \mathcal{F}(\mathbf{x}\mathbf{S}) \cdot \mathbf{T}$$

# Solving MCE [Reijnders, Samardjiska & T., 2022]

- reduction: an MCE instance $(k, n, m, \mathcal{C}, \mathcal{D})$ results in a QMLE instance $(m+n, k, \mathcal{F}, \mathcal{P})$
- solving the instance using a birthday-based algorithm $\mathcal{O}^*(q^{2/3(m+n)})$ [Bouillaguet, Fouque & Véber, 2013]

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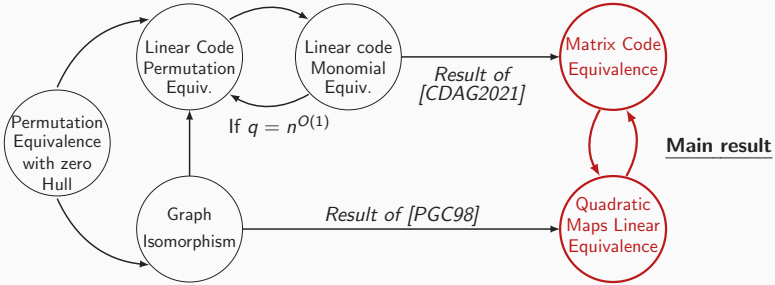


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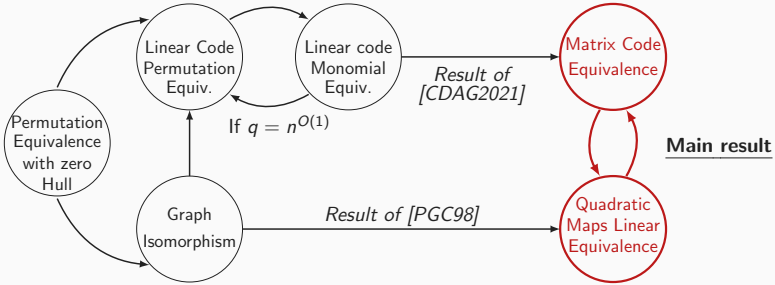


- Main result: MCE **is equivalent to** QMLE
- Gives **improved upper bound** to complexity of solving MCE (w.l.o.g. assume $m \leqslant n$)
  - solvable in $\mathcal{O}^*(q^{2/3(m+n)})$ time, when $k \leqslant n + m$ can be improved to $\mathcal{O}^*(q^m)$

# Matrix code equivalence: a cryptographic group action?

# MCE as cryptographic group action.

$$\mu : \mathcal{C} \to \mathcal{D}$$
$$\mathbf{C} \mapsto \mathbf{ACB}$$

- $\mu$ can be seen as element $(\mathbf{A}, \mathbf{B}) \in \mathrm{GL}_m(q) \times \mathrm{GL}_n(q)$

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- hence, $\mathrm{GL}_m(q) \times \mathrm{GL}_n(q)$ acts on $k$-dimensional matrix codes $\mathcal{C} \subset \mathcal{M}_{m \times n}(\mathbb{F}_q)$.
- one-way: our analysis show that MCE is hard.

# Building crypto from group actions

**Cryptographic Group Action:** $G \times X \to X$

Given $x_1$ and $x_2$, it is hard to find an element $g$ s.t. $x_2 = g \cdot x_1$

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- **Zero-Knowledge Interactive Proof of knowledge**
  - Zero-Knowledgness
  - soundness
  - can be used as identification scheme (IDS)

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- **Digital Signature via Fiat-Shamir transform**
  - F-S is a common strategy for PQ signatures
    - Dilithium, MQDSS, Picnic in NIST competition
  - From cryptographic group actions
    - Patarin's signature, LESS-FM, CSIDH, SeaSign ...

# Zero-Knowledge Interactive Proof of knowledge from group actions

Let $g$ be an element s.t. $x_1 = g \cdot x_0$.

Given $x_0, x_1$, the prover $\mathcal{P}$ wants to prove to the verifier $\mathcal{V}$ knowledge of $g$ without revealing any information about it


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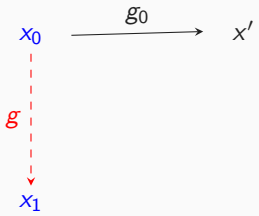


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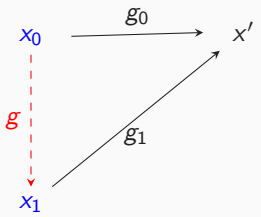


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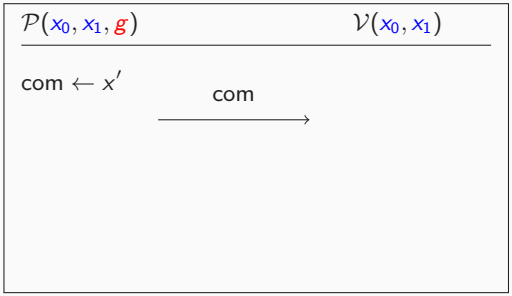

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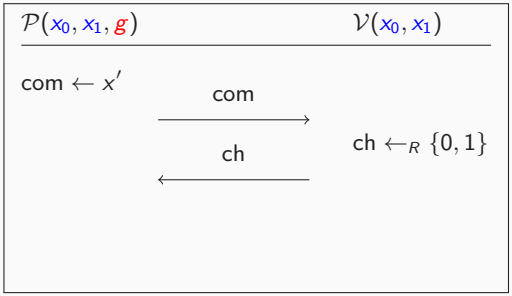

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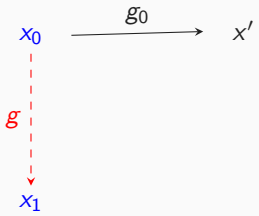


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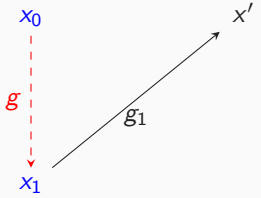


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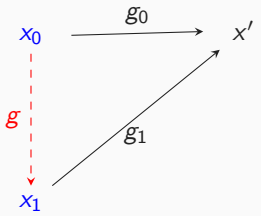


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# Digital Signatures via the Fiat-Shamir transform

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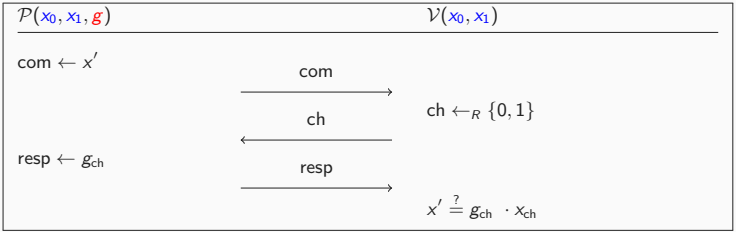

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FS signature

| **Signer**(pk, sk) | **Verifier**(pk) |
|---|---|
| $\text{com} \leftarrow (x', x'', \ldots, x^{(r)})$ | |
| $\text{ch} \leftarrow H(m, \text{com})$ | $\text{ch} \leftarrow H(m, \text{com})$ |
| $\text{resp} \leftarrow (g_{\text{ch}_1}, g_{\text{ch}_2}, \ldots, g_{\text{ch}_r})$ | $b \leftarrow \text{Vf}(\text{pk}, \text{com}, \text{ch}, \text{resp})$ |
| **output** : $\sigma = (\text{com}, \text{resp})$ | **output** : $b$ |

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(5) We construct (linkable) ring signatures